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THE PROPERTIES OF ENTIRE FUNCTIONS OF BOUNDED VALUE L-DISTRIBUTION IN DIRECTION

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We generalize a concept of function of bounded value l-distribution for multidimensional case. Besides we obtain a connection between class of entire functions of bounded value L-distribution in direction and entire functions of bounded L-index in direction.

Keywords: *entire function, bounded L-index in direction, bounded value L-distribution in direction, directional derivative.*

Entire functions of bounded value distribution and of bounded value l-distribution are investigated in the papers [1]-[3]. Particularly there are proved a connection between these classes of functions and classes of entire functions of bounded index and bounded l-index in papers [1] and [3]. We introduced the concept entire function of bounded L-index in direction in [4]. In connection with these papers we put next question: is there generalization of concept of entire function of bounded value l-distribution for entire functions of several complex variables and is there a connection between this new class and functions of bounded L-index in direction?

Let $L(z)$, $z\in \mathbb{C}^n$, be a positive continuous function.

Definition 1 *(see [4]). An entire function of $F(z)$, $z\in\mathbb{C}^n$, is called function of bounded L-index in the direction of $\mathbf{b}\in\mathbb{C}^n$, if there exists $m_0\in\mathbb{Z}_+$ such that for $m\in\mathbb{Z}_+$ and every $z\in\mathbb{C}^n$ next inequality is true:*

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial\mathbf{b}^m}\right|\le\max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\right|:0\le k\le m_0\right\},$$

where $\frac{\partial^0 F(z)}{\partial\mathbf{b}^0}=F(z)$, $\frac{\partial F(z)}{\partial\mathbf{b}}=\sum_{j=1}^{n}\frac{\partial F(z)}{\partial z_j}b_j$, $\frac{\partial^k F(z)}{\partial\mathbf{b}^k}=\frac{\partial}{\partial\mathbf{b}}\left(\frac{\partial^{k-1}F(z)}{\partial\mathbf{b}^{k-1}}\right)$, $k\ge 2$.

We denote

$$\lambda_1^{\mathbf{b}}(z,t_0,\eta)=\inf\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}:|t-t_0|\le\frac{\eta}{L(z+t_0\mathbf{b})}\right\},$$

$$\lambda_2^{\mathbf{b}}(z,t_0,\eta)=\sup\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}:|t-t_0|\le\frac{\eta}{L(z+t_0\mathbf{b})}\right\},$$

$$\lambda_1^{\mathbf{b}}(z,\eta)=\inf\{\lambda_1^{\mathbf{b}}(z,t_0,\eta):t_0\in\mathbb{C}\},\lambda_2^{\mathbf{b}}(z,\eta)=\inf\{\lambda_2^{\mathbf{b}}(z,t_0,\eta):t_0\in\mathbb{C}\},$$

$\lambda_1^{\mathbf{b}}(\eta)=\inf\{\lambda_1^{\mathbf{b}}(z,\eta): z\in \mathbb{C}^n\}$, $\lambda_2^{\mathbf{b}}(\eta)=\inf\{\lambda_2^{\mathbf{b}}(z,\eta): z\in \mathbb{C}^n\}$.

A class of functions L, which satisfy the condition $0<\lambda_1^{\mathbf{b}}(\eta)\le\lambda_2^{\mathbf{b}}(\eta)<+\infty$ for all $\eta\le 0$, we denote by $Q_{\mathbf{b}}^n$.

For fixed $z^0\in\mathbb{C}^n$ let c_k^0 are zeros of function $F(z^0+t\mathbf{b})$, i. e. $F(z^0+c_k^0\mathbf{b})=0$. Then we denote $n(r,z^0,t_0,1/F)=\sum_{|c_k^0-t_0|\le r}1$ be a normed counting function of sequence zeros c_k^0.

Definition 2 *An entire function $F(z)$, $z\in\mathbb{C}^n$, is called function of bounded value L-distribution in direction $\mathbf{b}\in\mathbb{C}^n$ if exists $p\in\mathbb{C}$ $\forall z_0\in\mathbb{C}^n$ such that $F(z^0+t\mathbf{b})\not\equiv 0$, and $\forall t_0\in\mathbb{C}$ $\forall w\in\mathbb{C}$ next inequality is true* $n(1/L(z^0+t_0\mathbf{b}),z^0,t_0,1/F\quad w)\le p$,

i.e. the equation $F(z^0+t\mathbf{b})=w$ has in $\left\{t:|t-t_0|\le\frac{1}{L(z^0+t\mathbf{b})}\right\}$ at most p solutions and, thus, $F(z^0+t\mathbf{b})$ is p-valent in $\left\{t:|t-t_0|\le\frac{1}{L(z^0+t\mathbf{b})}\right\}$.

The corresponding Sheremeta's result [3] is generalized for entire functions of bounded value L-distribution in direction.

Theorem 1 *Let $L\in Q_{\mathbf{b}}^n$. Entire function $F(z)$, $z\in\mathbb{C}^n$, is a function of bounded value L-distribution in direction $b\in\mathbb{C}^n$ if and only if its directional derivative $\frac{\partial F}{\partial\mathbf{b}}$ is of bounded L-index in direction b.*

Proof. Suppose that F if of bounded value L-distribution in direction $\mathbf{b}$, i. e. for all $z^0\in\mathbb{C}^n$ such that $F(z^0+t\mathbf{b})\not\equiv 0$ and for all $t^0\in\mathbb{C}$ function $F(z^0+t\mathbf{b})$ is p-valent in each disc $\left\{t:|t-t_0|\le\frac{1}{L(z^0+t\mathbf{b})}\right\}$

To prove this theorem we need an following theorem ([5], p. 48, Theorem 2.8).

Theorem 2 *[5] Let $D_0=\{t:|t\quad t_0|<R\}$, $0<R<\infty$. If an analytic function in D_0 is p-valent in D_0 then for $j>p$*

$$\frac{|f^{(j)}(t_0)|}{j!}R^j\le(Aj)^{2p}\max\ \frac{|f^{(k)}(t_0)|}{k!}R^k:1\le k\le p\ ,\tag{1}$$

where $A\equiv$ const and $A\ge\max\limits_{j>p}\frac{p+2}{2}(8e^{\pi^2})^p\left(1-\frac{1}{j}\right)^j$.

By Theorem 2 inequality (1) holds with $R=\frac{1}{L(z^0+t_0\mathbf{b})}$ for function $F(z^0+t\mathbf{b})$ as function of one variable $t\in\mathbb{C}$ for every fixed $z^0\in\mathbb{C}^n$. Let $f(t)=F(z^0+t\mathbf{b})$, then we can easy prove that for every $m\in\mathbb{N}$ next equality is true $f^{(p)}(t)=\dfrac{\partial^p F(z^0+t\mathbf{b})}{\partial \mathbf{b}^p}$. Put $j=p+1$ in Theorem 2. Then from (1) we obtain

$$\frac{\left|\dfrac{\partial^{p+1}F(z^0+t_0\mathbf{b})}{\partial \mathbf{b}^{p+1}}\right|}{(p+1)!L^{p+1}(z_0+t_0\mathbf{b})}\le (A(p+1))^{2p}\max\left\{\frac{\left|\dfrac{\partial^{k}F(z^0+t_0\mathbf{b})}{\partial \mathbf{b}^{k}}\right|}{k!L^{k}(z_0+t_0\mathbf{b})}:1\le k\le p\right\}\Rightarrow$$

$$\frac{\left|\dfrac{\partial^{p+1}F(z^0+t_0\mathbf{b})}{\partial \mathbf{b}^{p+1}}\right|}{L^{p+1}(z_0+t_0\mathbf{b})}\le (p+1)!(A(p+1))^{2p}\max\left\{\frac{\left|\dfrac{\partial^{k}F(z^0+t_0\mathbf{b})}{\partial \mathbf{b}^{k}}\right|}{L^{k}(z_0+t_0\mathbf{b})}:1\le k\le p\right\}\cdot\max\left\{\frac{1}{k!}:\right.$$

$$\left.1\le k\le p\right\}\Rightarrow\frac{\left|\dfrac{\partial^{p}}{\partial \mathbf{b}^{p}}\dfrac{\partial F(z^0+t_0\mathbf{b})}{\partial \mathbf{b}}\right|}{L^{p}(z_0+t_0\mathbf{b})}\le$$

$$\le L(z^0+t_0\mathbf{b})\cdot(p+1)!A^{2p}(p+1)^{2p}\max\left\{\frac{\left|\dfrac{\partial^{k-1}}{\partial \mathbf{b}^{k-1}}\dfrac{\partial F(z^0+t_0\mathbf{b})}{\partial \mathbf{b}}\right|}{L^{k}(z_0+t_0\mathbf{b})}:\right.$$

$$\left.0\le k-1\le p-1\right\}\Rightarrow\frac{\left|\dfrac{\partial^{p}}{\partial \mathbf{b}^{p}}\dfrac{\partial F(z^0+t_0\mathbf{b})}{\partial \mathbf{b}}\right|}{L^{p}(z_0+t_0\mathbf{b})}\le$$

$$\le (p+1)!A^{2p}(p+1)^{2p}\max\left\{\frac{\left|\dfrac{\partial^{k-1}}{\partial \mathbf{b}^{k-1}}\dfrac{\partial F(z^0+t_0\mathbf{b})}{\partial \mathbf{b}}\right|}{L^{k-1}(z_0+t_0\mathbf{b})}:0\le k-1\le p-1\right\}.$$

Now we need a next analogues Hayman's theorem for entire functions of bounded L-index in direction ([4], Theorem 8).

Theorem 3 *([4]) Let* $L\in Q^n_{\mathbf{b}}$. *An entire function* $F(z)$, $z\in \mathbb{C}^n$, *is of bounded* L*-index in direction* $\mathbf{b}\in\mathbb{C}^n$ *if, and only if, there exists numbers* $p\in\mathbb{Z}_+$ *and* $C>0$ *such that for each* $z\in\mathbb{C}^n$

$$\frac{1}{L^{p+1}(z)}\left|\frac{\partial^{p+1}F(z)}{\partial \mathbf{b}^{p+1}}\right|\le C\max\left\{\frac{1}{L^{k}(z)}\left|\frac{\partial^{k}F(z)}{\partial \mathbf{b}^{k}}\right|:0\le k\le p\right\}\qquad(2)$$

Thus, for $\dfrac{\partial F}{\partial \mathbf{b}}$ inequality (2) holds with p 1 instead of p and with $C=(p+1)!A^{2p}(p+1)^{2p}$. In Theorem 2 constant $A\ge\max\limits_{j>p}\dfrac{p+2}{2}(8e^{\pi^2})^p(1-\dfrac{1}{j})^j$

is independent of z^0, because p is independent of z^0. Then $C=(p+1)!A^{2p}(p+1)^{2p}$ is independent of z^0. Thus by Theorem 3 $\frac{\partial F}{\partial \mathbf{b}}$ is of bounded L-index in direction $\mathbf{b}$.

On the contrary, let $\frac{\partial F}{\partial \mathbf{b}}$ be of bounded L-index in direction $\mathbf{b}\in \mathbb{C}^n$. By Theorem 3 there exists $p\in \mathbb{Z}_+$ and $C\geq 1$ such that for each $z\in \mathbb{C}^n$ the next inequality is true

$$\frac{1}{L^{p+1}(z)}\left|\frac{\partial^{p+1}F(z)}{\partial \mathbf{b}^{p+1}}\right|\leq C\max\left\{\frac{1}{L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right|:1\leq k\leq p\right\} \qquad (3)$$

We consider a disk $K_0=\left\{t\in \mathbb{C}:|t-t_0|\leq \frac{1}{L(z^0+t_0\mathbf{b})}\right\}$, $t_0\in \mathbb{C}$, $z^0\in \mathbb{C}^n$.

We remark that if $L(z)\in Q_{\mathbf{b}}^n$ and $z^0\in \mathbb{C}^n$, $t_0\in \mathbb{C}$ then for all $r>0$ the inequality $|t-t_0|\leq \frac{r}{L(z^0+t_0\mathbf{b})}$ and definition of class $Q_{\mathbf{b}}^n$ imply the inequality

$$\lambda_1^{\mathbf{b}}(r)L(z^0+t_0\mathbf{b})\leq L(z^0+t\mathbf{b})\leq \lambda_2^{\mathbf{b}}(r)L(z^0+t_0\mathbf{b}). \qquad (4)$$

From (3) and (4) we have

$$\frac{1}{(p+1)!}\left|\frac{\partial^{p+1}F(z^0+t\mathbf{b})}{\partial \mathbf{b}^{p+1}}\right|\left(\frac{1}{C\lambda_2^{\mathbf{b}}(1)L(z^0+t_0\mathbf{b})}\right)^{p+1}\leq \frac{Cp!}{(p+1)!}\max\left\{\frac{1}{k!}\left|\frac{\partial^k F(z^0+t\mathbf{b})}{\partial \mathbf{b}^k}\right|\times\right.$$

$$\left.\times\left(\frac{1}{C\lambda_2^{\mathbf{b}}(1)L(z^0+t_0\mathbf{b})}\right)^k\left(\frac{L(z^0+t\mathbf{b})}{C\lambda_2^{\mathbf{b}}(1)L(z^0+t_0\mathbf{b})}\right)^{p+1-k}:1\leq k\leq p\right\}\leq$$

$$\leq\frac{C}{p+1}\max\left\{\frac{1}{k!}\left|\frac{\partial^k F(z^0+t\mathbf{b})}{\partial \mathbf{b}^k}\right|\left(\frac{1}{C\lambda_2^{\mathbf{b}}(1)L(z^0+t_0\mathbf{b})}\right)^k\left(\frac{1}{C}\right)^{p+1-k}:1\leq k\leq p\right\}\leq$$

$$\leq\max\left\{\frac{1}{k!}\left|\frac{\partial^k F(z^0+t\mathbf{b})}{\partial \mathbf{b}^k}\right|\left(\frac{1}{C\lambda_2^{\mathbf{b}}(1)L(z^0+t_0\mathbf{b})}\right)^k:1\leq k\leq p\right\}. \qquad (5)$$

To prove this theorem we need an following theorem ([5], p. 44, Theorem 2.7).

Theorem 4 *Let* $D_0=\{t\in\mathbb{C}:|t-t_0|<R\}$, $0<R<+\infty$, *and* $f(t)$ *be analytic function in* D_0. *If for all* $z\in D_0$

$$\left(\frac{R}{2}\right)^{p+1}\frac{|f^{(p+1)}(t)}{(p+1)!}\leq\max\left\{\left(\frac{R}{2}\right)\frac{|f^{(k)}(z)|}{k!}:1\leq k\leq p\right\} \qquad (6)$$

then $f(t)$ is p-valent in $\left\{t\in\mathbb{C}:|t-t_0|\le\frac{R}{25\sqrt{p+1}}\right\}$, i. e. $f(t)$ assumes each values at most p times.

The inequality (5) implies inequality (6) with $R=\frac{2}{C\lambda_2^{\mathbf{b}}(1)L(z^0+t_0\mathbf{b})}$. By Theorem 4 the function $F(z^0+t\mathbf{b})$ is p-valent in the disk $\{t\in\mathbb{C}:|t\quad t_0|\le\frac{\rho}{L(z^0+t_0\mathbf{b})}\}$, $\rho=\frac{2}{25C\lambda_2^{\mathbf{b}}(1)\sqrt{p+1}}$.

Let t_j be arbitrary point in K_0 and $K_j^*=\{t\in\mathbb{C}:|t-t_i|\le\frac{\rho}{L(z^0+t_j\mathbf{b})}\}$. Since

$$L(z^0+t_j\mathbf{b})\le\lambda_2^{\mathbf{b}}(1)L(z^0+t_0\mathbf{b})$$

from definitions class $Q_{\mathbf{b}}^n$, we see that

$$K_j=\{t\in\mathbb{C}:|t-t_j|\le\frac{\rho}{\lambda_2^{\mathbf{b}}(1)L(z^0+t_0\mathbf{b})}\}\subset K_j^*.$$

We can repeat the above considerations to the set $\left\{t\in\mathbb{C}:|t-t_j|\le\frac{1}{L(z^0+t_j\mathbf{b})}\right\}$ and as above we obtain that $F(z^0+t\mathbf{b})$ is p-valent in K_j^*. But $K_j\subset K_j^*$ therefore $F(z^0+t\mathbf{b})$ is p-valent in K_j.

Finally we remark that each closed discs of radius R_* we can cover by a finite number m_* of closed disks of radius $\rho_*<R_*$ and with center in this disk, moreover, $m_*<B_*(R_*/\rho_*)^2$, where $B_*>0$ is an absolute constant. Hence, K_0 can be covered be a finite number m of disks K_j, where $m\le 625(p+1)C^2(\lambda_2^{\mathbf{b}}(1))^2/4$. Since $F(z^0+t\mathbf{b})$ in K_j is p-valent, it is mp-valent in K_0.

In view of arbitrarity of t_0 and z^0, the theorem is proved.

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ВЛАСТИВОСТІ ЦІЛИХ ФУНКЦІЙ ОБМЕЖЕНОГО L-РОЗПОДІЛУ ЗНАЧЕНЬ ЗА НАПРЯМКОМ

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Узагальнено поняття функції обмеженого L-розподілу значень на багатовимірний випадок. Отримано зв'язок між класом цілих функцій обмеженого L-розподілу значень за напрямком та класом цілих функцій обмеженого L-індексу за напрямком.

Ключові слова: *ціла функція, обмежений L-індекс за напрямом, обмежений L-розподіл значень за напрямком, похідна за напрямком.*