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# SUPERSYMMETRIC 2-HOMOGENEOUS POLYNOMIALS ON $L_2((-\infty,+\infty))$

**Yu.S. Sharyn**
*Vasyl Stefanyk Carpathian National University;*
*76018, 57 Shevchenka street, Ivano-Frankivsk, Ukraine;*
*e-mail: yurii.sharyn.21@pnu.edu.ua*

*The work is devoted to the study of supersymmetric continuous 2-homogeneous $\mathbb{K}$-valued, where $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, polynomials on the Hilbert space $L_2((-\infty,+\infty))$ of all functions $x:(-\infty,+\infty)\to\mathbb{K}$ such that $x^2$ is Lebesgue integrable. We show that every such a polynomial $P$ can be represented as $P(x)=\alpha\left(\int_0^{+\infty}x^2(t)dt-\int_{-\infty}^0 x^2(t)dt\right)$, where $\alpha\in\mathbb{K}$. Consequently, the vector space of all such polynomials is one-dimensional.*

***Key words:*** *polynomial, symmetric function, supersymmetric function, Hilbert space, Lebesgue integrable function.*

## 1. Introduction

The theory of supersymmetric polynomials on sequence Banach spaces has many applications, especially, in statistical quantum physics (see [1, 3]). In this work is consider the notion of supersymmetry of polynomials on some function space. Namely, we investigate supersymmetric continuous polynomials on the Hilbert space of Lebesgue measurable functions on the real axis, squares of which are Lebesgue integrable. Both real and complex cases are considered. We establish the structure of a supersymmetric continuous 2-homogeneous scalar-valued polynomial on this Hilbert space. We show that the vector space of all such polynomials is one-dimensional.

*Corresponding author

## 2. Preliminaries

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let us denote by $\chi_A$ the characteristic function of a set $A \subset (-\infty, +\infty)$.

*The space* $L_2(M)$. Let $L_2(M)$, where $M$ is a measurable subset of $(-\infty, +\infty)$ with the positive measure, be the Hilbert space over $\mathbb{K}$ consisting of all Lebesgue measurable functions $x : M \to \mathbb{K}$ such that $x^2$ is Lebesgue integrable over $M$, with norm

$$\|x\| = \left( \int_M |x(t)|^2 \, dt \right)^{1/2}.$$

*Symmetric 2-homogeneous polynomials on* $L_2([0, +\infty))$. Let $\Xi_{[0,+\infty)}$ be the set of all bijections $\sigma : [0, +\infty) \to [0, +\infty)$ that preserve the Lebesgue measure, i.e., $\sigma^{-1}(A)$ is measurable and $\mu(\sigma^{-1}(A)) = \mu(A)$ for every measurable $A \subset [0, +\infty)$, where $\mu$ is the Lebesgue measure. A function $f : L_2([0, +\infty)) \to \mathbb{K}$ is called symmetric if $f(x \circ \sigma) = f(x)$ for every $x \in L_2([0, +\infty))$ and $\sigma \in \Xi_{[0,+\infty)}$.

The following theorem is the partial result of [2, Theorem 2.12] for the case $\mathbb{K} = \mathbb{R}$ and [4, Theorem 3] for the case $\mathbb{K} = \mathbb{C}$.

**Theorem 2.1.** *Let* $Q : L_2([0, +\infty)) \to \mathbb{K}$ *be a symmetric continuous 2-homogeneous polynomial. Then*

$$Q(x) = \alpha \int_0^{+\infty} (x(t))^2 \, dt$$

*for every* $x \in L_2([0, +\infty))$, *where* $\alpha = Q(\chi_{(0,1)})$.

## 3. The main result

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\mathscr{X} = L_2((-\infty, +\infty))$ over $\mathbb{K}$. Denote by $\Sigma_+$ the set of all bijections $\sigma_+ : (-\infty, +\infty) \to (-\infty, +\infty)$ preserving the Lebesgue measure such that $\sigma_+(t) = t$ for every $t \leq 0$. Analogously, denote by $\Sigma_-$ the set of all measure-preserving bijections $\sigma_- : (-\infty, +\infty) \to (-\infty, +\infty)$ such that $\sigma_-(t) = t$ for every $t \geq 0$.

For $Y \in \mathscr{X}$ and $k > 0$ we define the operator $F_{Y,k} : \mathscr{X} \to \mathscr{X}$ by

$$(F_{Y,k}x)(t) = \begin{cases} x(t-k), & \text{if } t > k, \\ x(t+k), & \text{if } t < -k, \\ Y(t), & \text{if } 0 \le t \le k, \\ Y(-t), & \text{if } -k \le t < 0. \end{cases}$$

A functional $H : \mathscr{X} \to \mathbb{K}$ is called ***supersymmetric*** if for all $x, Y \in \mathscr{X}$, all $k > 0$, all $\sigma_+ \in \Sigma_+$ and all $\sigma_- \in \Sigma_-$ the following equalities hold:

$$\begin{aligned} &1.\quad H(x) = H(x \circ \sigma_+), \\ &2.\quad H(x) = H(x \circ \sigma_-), \\ &3.\quad H(x) = H(F_{Y,k}x). \end{aligned}$$

**Lemma 3.1.** *Let $H : \mathscr{X} \to \mathbb{K}$ be a supersymmetric functional. Then for all $x_1, \ldots, x_n, Y \in \mathscr{X}$, $a_1, \ldots, a_n \in \mathbb{K}$ and $k > 0$,*

$$H\left(\sum_{i=1}^{n} a_i F_{Y,k} x_i\right) = H\left(\sum_{i=1}^{n} a_i x_i\right).$$

***Proof.*** For $t \in \mathbb{R}$ we obtain

$$\sum_{i=1}^{n} a_i (F_{Y,k} x_i)(t) = \begin{cases} \sum_{i=1}^{n} a_i x_i(t-k), & \text{if } t > k, \\ \sum_{i=1}^{n} a_i x_i(t+k), & \text{if } t < -k, \\ (\sum_{i=1}^{n} a_i) Y(t), & \text{if } 0 \le t \le k, \\ (\sum_{i=1}^{n} a_i) Y(-t), & \text{if } -k \le t < 0. \end{cases}$$

Hence, $\sum_{i=1}^{n} a_i F_{Y,k} x_i = F_{(\sum_{i=1}^{n} a_i) Y,k} (\sum_{i=1}^{n} a_i x_i)$. By the supersymmetry property 3, $H(\sum_{i=1}^{n} a_i x_i) = H\left(F_{(\sum_{i=1}^{n} a_i) Y,k} (\sum_{i=1}^{n} a_i x_i)\right)$. Therefore,

$$H\left(\sum_{i=1}^{n} a_i F_{Y,k} x_i\right) = H\left(\sum_{i=1}^{n} a_i x_i\right).$$

This completes the proof. □

Let us investigate supersymmetric 2-homogeneous polynomials on $\mathscr{X}$. For a 2-homogeneous polynomial $P : \mathscr{X} \to \mathbb{K}$ we will denote by $A_P$ the associated to $P$ symmetric bilinear form. Note that $A_P$ can be obtained as

$$A_P(x,y) = \frac{1}{2}\big(P(x+y) - P(x) - P(y)\big). \tag{1}$$

**Lemma 3.2.** *Let $P : \mathscr{X} \to \mathbb{K}$ be a supersymmetric 2-homogeneous polynomial. Then the following properties hold for all $x, y, Y \in \mathscr{X}$, $k > 0$, $\sigma_+ \in \Sigma_+$, $\sigma_- \in \Sigma_-$:*

$$\begin{aligned} &\textit{(i)} && A_P(x,y) = A_P(x\circ\sigma_+, y\circ\sigma_+),\\ &\textit{(ii)} && A_P(x,y) = A_P(x\circ\sigma_-, y\circ\sigma_-),\\ &\textit{(iii)} && A_P(x,y) = A_P(F_{Y,k}x, F_{Y,k}y). \end{aligned}$$

***Proof.*** We prove (i)–(iii) using the corresponding supersymmetry properties of $P$.

**(i)** Using $P(x) = P(x\circ\sigma_+)$, we have

$$A_P(x\circ\sigma_+, y\circ\sigma_+) = \tfrac{1}{2}\Big(P(x\circ\sigma_+ + y\circ\sigma_+) - P(x\circ\sigma_+) - P(y\circ\sigma_+)\Big) =$$
$$= \tfrac{1}{2}\Big(P((x+y)\circ\sigma_+) - P(x\circ\sigma_+) - P(y\circ\sigma_+)\Big) = \tfrac{1}{2}\Big(P(x+y) - P(x) - P(y)\Big) =$$
$$= A_P(x,y).$$

**(ii)** The proof is identical using $P(x) = P(x\circ\sigma_-)$.

**(iii)** Using $P(x) = P(F_{Y,k}x)$ and Lemma 3.1, we obtain

$$A_P(F_{Y,k}x, F_{Y,k}y) = \tfrac{1}{2}\Big(P(F_{Y,k}x + F_{Y,k}y) - P(F_{Y,k}x) - P(F_{Y,k}y)\Big) =$$
$$= \tfrac{1}{2}\Big(P(x+y) - P(x) - P(y)\Big) = A_P(x,y).$$

This completes the proof. ☐

**Lemma 3.3.** *Let $P : \mathscr{X} \to \mathbb{K}$ be a supersymmetric 2-homogeneous polynomial. For intervals $I \subset (-\infty, 0)$ and $J \subset (0, \infty)$, $A_P(\chi_I, \chi_J) = 0$.*

***Proof.*** Let $x := \chi_{(-|I|,0)}$ and $y := \chi_{(0,|J|)}$, where $|I|$, $|J|$ denote lengths of the intervals $I$, $J$, respectively. By properties $(i)$, $(ii)$ (see Lemma 3.2) of the form $A_P$, we have $A_P(\chi_I,\chi_J) = A_P(x,y)$. Without loss of generality, we assume that $|J| \leqslant |I|$. By property $(iii)$ (see Lemma 3.2) of the form $A_P$, we get

$$A_P(\chi_{(|I|-|J|,0)},0) = A_P(F_{\chi_J,|J|}(\chi_{(|I|-|J|,0)}),F_{\chi_J,|J|}(0)) = A_P(x,y).$$

Since $A_P$ is bilinear, we have $A_P(\chi_{(|I|-|J|,0)},0) = 0$, which completes the proof. □

**Corollary 3.1.** *Let $P : \mathscr{X} \to \mathbb{K}$ be a supersymmetric 2-homogeneous polynomial. Then $P(\chi_{(-1,0)}) = -P(\chi_{(0,1)})$.*

***Proof.*** By the supersymmetry property 3, where we set $x=0$, $Y=\chi_{[0,1)}$, and $k=1$, we have $P(0) = P(\chi_{(-1,1)})$. Therefore, since $P(0) = 0$, it follows that $P(\chi_{(-1,1)}) = 0$. On the other hand, by the binomial formula,

$$P(\chi_{(-1,1)})=P(\chi_{(-1,0)}+\chi_{(0,1)})=P(\chi_{(-1,0)})+2A_P(\chi_{(-1,0)},\chi_{(0,1)})+P(\chi_{(0,1)}).$$

By Lemma 3.3, $A_P(\chi_{(-1,0)},\chi_{(0,1)}) = 0$. Therefore $P(\chi_{(-1,0)}) = -P(\chi_{(0,1)})$. □

**Corollary 3.2.** *Let $P : \mathscr{X} \to \mathbb{K}$ be a supersymmetric 2-homogeneous polynomial. For $x = \sum_{i=1}^{n} \alpha_i \chi_{B_i}$ and $y = \sum_{j=1}^{m} \beta_j \chi_{C_j}$, where $\alpha_i, \beta_j \in \mathbb{K}$, $B_i, C_j$ are intervals such that $B_i \subset (-\infty,0)$, $C_j \subset (0,+\infty)$, we have $A_P(x,y) = 0$.*

***Proof.*** By the bilinearity of $A_P$, $A_P(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i\beta_j A_P(\chi_{B_i},\chi_{C_j})$. By Lemma 3.2, for every $i,j$, $A_P(\chi_{B_i},\chi_{C_j}) = 0$. Hence, $A_P(x,y) = 0$. □

**Corollary 3.3.** *Let $P : \mathscr{X} \to \mathbb{K}$ be a continuous supersymmetric 2-homogeneous polynomial. Let $x,y \in \mathscr{X}$ be such that $x|_{(0,+\infty)} \overset{a.e.}{=} 0$ and $y|_{(-\infty,0)} \overset{a.e.}{=} 0$. Then $A_P(x,y) = 0$.*

***Proof.*** Since $x,y \in \mathscr{X}$, $x|_{(0,+\infty)} \overset{\text{a.e.}}{=} 0$, and $y|_{(-\infty,0)} \overset{\text{a.e.}}{=} 0$, it follows that there exist sequences $\{x_k\}_{k=1}^{\infty}, \{y_l\}_{l=1}^{\infty} \subset \mathscr{X}$ such that

$$\lim_{k\to\infty} x_k = x, \quad \lim_{l\to\infty} y_l = y, \text{ and}$$

$$x_k = \sum_{i=1}^{n_k} \alpha_i^{(k)} \chi_{B_i^{(k)}}, \quad y_l = \sum_{j=1}^{m_l} \beta_j^{(l)} \chi_{C_j^{(l)}},$$

where $\alpha_i^{(k)}, \beta_j^{(l)} \in \mathbb{K}$, and $B_i^{(k)}, C_j^{(l)}$ are intervals such that $B_i^{(k)} \subset (-\infty, 0)$, $C_j^{(l)} \subset (0, +\infty)$. Since $P$ is continuous, it follows that $A_P$ is continuous. Therefore,

$$A_P(x,y) = \lim_{k\to\infty} \lim_{l\to\infty} A_P(x_k, y_l).$$

By Corollary 3.2, $A_P(x_k, y_l) = 0$. Consequently, $A_P(x,y) = 0$. □

**Theorem 3.1.** *Let $P : \mathscr{X} \to \mathbb{K}$ be a continuous supersymmetric 2-homogeneous polynomial. Then*

$$P(x) = \alpha \left( \int_0^{+\infty} (x(t))^2 \, dt - \int_{-\infty}^{0} (x(t))^2 \, dt \right)$$

*for every $x \in \mathscr{X}$, where $\alpha = P(\chi_{[0,1]})$.*

***Proof.*** Let $x \in \mathscr{X}$. Let $x_+, x_- \in \mathscr{X}$ be defined by

$$x_+(t) = x(t)\chi_{(0,+\infty)}(t), \quad x_-(t) = x(t)\chi_{(-\infty,0)}(t)$$

for $t \in (-\infty, +\infty)$. By the binomial formula,

$$P(x) = P(x_+ + x_-) = P(x_+) + 2A_P(x_+, x_-) + P(x_-).$$

By Corollary 3.3, $A_P(x_+, x_-) = 0$. Therefore,

$$P(x) = P(x_+) + P(x_-). \tag{2}$$

Let operators $\imath_+, \imath_- : L_2([0,+\infty)) \to \mathscr{X}$ be defined by

$$\imath_+(z) = \begin{cases} z(t), & \text{if } t > 0, \\ 0, & \text{if } t \le 0, \end{cases} \quad \text{and} \quad \imath_-(z) = \begin{cases} z(-t), & \text{if } t < 0, \\ 0, & \text{if } t \ge 0 \end{cases} \tag{3}$$

resp., where $z \in L_2([0,+\infty))$. It can be checked that both $\imath_+$ and $\imath_-$ are linear and continuous. Therefore $P \circ \imath_+$ and $P \circ \imath_-$ are continuous

2-homogeneous polynomials on $L_2([0,+\infty))$. By the properties 1 and 2 of the supersymmetry of $P$, $P\circ\iota_+$ and $P\circ\iota_-$ are symmetric. Therefore, by Theorem 2.1,

$$(P\circ\iota_+)(z)=\alpha\int_0^{+\infty}(z(t))^2\,dt \quad\text{and}\quad (P\circ\iota_-)(z)=\beta\int_0^{+\infty}(z(t))^2\,dt \tag{4}$$

for every $z\in L_2([0,+\infty))$, where

$$\alpha=(P\circ\iota_+)(\chi_{(0,1)}) \quad\text{and}\quad \beta=(P\circ\iota_-)(\chi_{(0,1)}),$$

i.e., taking into account (3) and Corollary 3.1,

$$\alpha=P(\chi_{(0,1)}) \quad\text{and}\quad \beta=P(\chi_{(-1,0)})=-P(\chi_{(0,1)}). \tag{5}$$

Let $z_1,z_2\in L_2([0,+\infty))$ be defined by $z_1(t)=x(t)$ and $z_2(t)=x(-t)$, resp. for $t\in[0,+\infty)$. It can be checked that $\iota_+(z_1)=x_+$ and $\iota_-(z_2)=x_-$. Therefore, by (4),

$$P(x_+)=P(\iota_+(z_1))=\alpha\int_0^{+\infty}(x(t))^2\,dt \text{ and}$$

$$P(x_-)=P(\iota_-(z_2))=\beta\int_0^{+\infty}(x(-t))^2\,dt=\beta\int_{-\infty}^0(x(t))^2\,dt$$

Therefore, taking into account (2) and (5), the desired equality holds. □

**Corollary 3.4.** *The vector space of all continuous supersymmetric 2-homogeneous polynomials on $\mathscr{X}$ is one-dimensional.*

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### *References*

1. Chernega I., Martsinkiv M., Vasylyshyn T., Zagorodnyuk A., *Applications of Supersymmetric Polynomials in Statistical Quantum Physics*, Quantum Reports **5** (4) (2023), 683–697. doi:10.3390/quantum5040043

2. González M., Gonzalo R., Jaramillo J. A., *Symmetric polynomials on rearrangement invariant function spaces*, J. Lond. Math. Soc. **59** (2) (1999), 681–697. doi:10.1112/S0024610799007164
3. Hryniv R., Kravtsiv V., Vasylyshyn T., Zagorodnyuk A., *Symmetric and supersymmetric polynomials on $\ell_p$ and partition functions in quantum statistical physics,* Physica Scripta **100** (7) (2025), Article number 075208. doi:10.1088/1402-4896/adde1e
4. Vasylyshyn T.V. *Symmetric polynomials on the Cartesian power of $L_p$ on the semi-axis*, Mat. Stud. **50** (1) (2018), 93–104. doi:10.15330/ms.50.1.93-104

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# СУПЕРСИМЕТРИЧНІ 2-ОДНОРІДНІ ПОЛІНОМИ НА ПРОСТОРІ $L_2((-\infty,+\infty))$

**Ю.С. Шарин**
*Карпатський національний університет імені Василя Стефаника;*
*76018, вул. Шевченка, 57, Івано-Франківськ, Україна;*
*e-mail: yurii.sharyn.21@pnu.edu.ua*

*Роботу присвячено дослідженню суперсиметричних неперервних 2-однорідних $\mathbb{K}$-значних, де $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, поліномів на гільбертовому просторі $L_2((-\infty,+\infty))$ всіх функцій $x:(-\infty,+\infty)\to\mathbb{K}$, для яких функція $x^2$ є інтегровною за Лебегом. Показано, що кожен такий поліном $P$ зображається як6 $P(x)=\alpha\left(\int_0^{+\infty} x^{(}t)dt - \int_{-\infty}^0 x^2(t)dt\right)$, де $\alpha\in\mathbb{K}$. Як наслідок, лінійний простір всіх таких поліномів є одновимірним.*

***Ключові слова:*** *поліном, симетрична функція, суперсиметрична функція, гільбертів простір, інтегровна за Лебегом функція.*