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ON THE PRESERVATION OF THE BOUNDEDNESS OF THE L-INDEX UNDER THE ACTION OF THE BERNARDI INTEGRAL OPERATOR AND THE RUSCHEWEYH DERIVATIVE

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We study the action of the Bernardi integral operator and the Ruscheweyh derivative on analytic functions of bounded l-index in a disc. Sufficient conditions are given for functions of bounded l-index under which their images under the action of the Bernardi integral operator, as well as the Rusheweyh derivative operator, are also functions of bounded l-index with the same function l. The proof is based on the theorem of M.M. Sheremeta and Z.M. Sheremeta (1999), which contains a sufficient condition for the boundedness of the l-index of an analytic function in a disc. It is formulated in the form of a restriction on the Taylor coefficients of a given analytic function of bounded l-index.

Key words: *univalent function; bounded l-index; Bernardi integral operator; Ruscheweyh derivative.*

1. Introduction.

It is known that there are many linear operators acting on the class of univalent functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ that do not take the functions outside the class of univalent functions, that is, the images of univalent functions are also univalent functions (see, for example, [1, 2]). On the one hand, functions from the class of functions of bounded index (or of bounded l-index) have a number of properties that are very close to the properties of univalent functions ([3–5]). For example, functions of bounded index (or of bounded l-index) are locally finitely-valent, i.e. they have bounded valued distribution. In this connection, Prof. O.B. Skaskiv suggested [6] that operators that preserve univalency may turn out to be those that preserve boundedness of the index (or boundedness of the l-index). It was formulated in the form of the following problem and hypothesis in [6]:

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Problem 8 in [6]. *What are operators preserving index boundedness for analytic functions?*

Conjecture 3 in [6]. *Differential operators preserving univalence also must preserve index boundedness for some additional assumptions.*

On the other hand, for example, the product of two functions of bounded index is always a function of bounded index, i.e., the multiplication operator on the fixed function of bounded index maps functions of bounded index onto functions of bounded index. But the product of two univalent functions, obviously, does not have to be a univalent function. That is, it cannot be stated, in general, that the image of every analytic function of bounded l-index is a function of bounded l-index when the given fixed function that defines the operator is only univalent. Finally, we can consider the following question: what is the class of univalent functions that define the multiplication operator, and the image of every function of bounded l-index is a function of bounded index? The class of such univalent functions, in view of the above, is nonempty and consists of those analytic functions that are both univalent and functions of bounded l-index. But we cannot say anything more specific about the class of such univalent functions at the moment.

Note that the notion of an analytic function of bounded index is meaningful only in the case of entire functions. On the other hand, the notion of a function of bounded l-index becomes meaningful already in the case of analytic functions in the unit disc, which is a natural domain in which the properties of various classes of univalent functions are investigated.

In this article, we obtain partial positive answer to conjecture of O. B. Skaskiv for two operators: the Bernardi integral operator and the Ruscheweyh derivative operator.

2. Definitions and auxiliary statement: analytic functions of bounded l-index.

Let

$$\mathbb{D}_R := \{z \in \mathbb{C} : \ |z| < R\}, \qquad R > 0,$$

and $l\colon [0,R) \to (0,+\infty)$ be a continuous function.

Definition 2.1. An analytic function f in $\mathbb{D}_R$ is said to be of *bounded l-index* if there exists an integer $N \geqslant 0$ such that

$$\frac{|f^{(n)}(z)|}{n!\,l(|z|)^n} \leqslant \max\left\{\frac{|f^{(k)}(z)|}{k!\,l(|z|)^k} : \ 0 \leqslant k \leqslant N\right\} \qquad (n \geqslant 0, \ z \in \mathbb{D}_R).$$

The smallest of these N is called the *l-index of the function f* in $\mathbb{D}_R$ and is denoted as $N(f,l;\mathbb{D}_R)$.

The following theorem from [7] indicates conditions on the Taylor coefficients of an analytic function in a disk which are sufficient for its l-index boundedness in this disk.

Theorem 2.1 (Sufficient condition). *Let*

$$f(z)=\sum_{m=j}^{\infty} a_m z^m, \qquad a_j\neq 0, \qquad j\geqslant 1,$$

be analytic in $\mathbb{D}_R$, and define

$$\alpha_j(R):=\sum_{n=1}^{\infty}\frac{(n+j)!}{j!\,n!}\frac{|a_{n+j}|}{|a_j|}R^n.$$

If $\alpha_j(R)<1$, then f is of bounded l-index in $\mathbb{D}_R$ for

$$l(r)=\frac{1}{R-r}, \qquad 0\leqslant r<R.$$

Moreover,

$$N(f,l;\mathbb{D}_R)\leqslant j+J(R),$$

where

$$J(R):=\min\left\{k\in\mathbb{Z}_+:\ k\geqslant j,\ \frac{1+\alpha_j(R)}{1-\alpha_j(R)}\leqslant\frac{(k+j)!}{k!\,j!}\right\}.$$

3. Bernardi integral operator.

For $\gamma>-1$, the Bernardi integral operator is defined by

$$(\mathscr{B}_\gamma f)(z):=\frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1}f(t)\,dt.$$

If

$$f(z)=\sum_{m=j}^{\infty} a_m z^m,$$

then

$$(\mathscr{B}_\gamma f)(z)=\sum_{m=j}^{\infty} b_m z^m, \qquad b_m=\frac{\gamma+1}{m+\gamma}a_m,$$

see [2].

Proposition 3.1. *Let*

$$f(z)=\sum_{m=j}^{\infty} a_m z^m, \qquad a_j\neq 0, \qquad j\geqslant 1,$$

be analytic in $\mathbb{D}_R$*, and assume that*

$$\alpha_j(R)=\sum_{n=1}^{\infty}\frac{(n+j)!}{j!\,n!}\,\frac{|a_{n+j}|}{|a_j|}\,R^n<1.$$

Then, for every $\gamma>-1$*, the function* $\mathscr{B}_\gamma f$ *is of bounded* l*-index in* $\mathbb{D}_R$ *for the same weight*

$$l(r)=\frac{1}{R-r}.$$

More precisely,

$$N(\mathscr{B}_\gamma f,l;\mathbb{D}_R)\leqslant j+J(R),$$

where $J(R)$ *is the quantity from Theorem 2.1.*

Proof. Put $F:=\mathscr{B}_\gamma f$. Since $j\geqslant 1$ and $\gamma>-1$, we have $j+\gamma>0$, so the first nonzero coefficient of F is still the coefficient of z^j. Therefore

$$\alpha_j^F(R)=\sum_{n=1}^{\infty}\frac{(n+j)!}{j!\,n!}\,\frac{|b_{n+j}|}{|b_j|}\,R^n.$$

Using the coefficient formula for b_m, we get

$$\frac{|b_{n+j}|}{|b_j|}=\frac{\gamma+1}{n+j+\gamma}\cdot\frac{j+\gamma}{\gamma+1}\,\frac{|a_{n+j}|}{|a_j|}=\frac{j+\gamma}{n+j+\gamma}\,\frac{|a_{n+j}|}{|a_j|}.$$

Hence,

$$\alpha_j^F(R)=\sum_{n=1}^{\infty}\frac{(n+j)!}{j!\,n!}\,\frac{j+\gamma}{n+j+\gamma}\,\frac{|a_{n+j}|}{|a_j|}\,R^n.$$

Since

$$0<\frac{j+\gamma}{n+j+\gamma}\leqslant 1 \qquad (n\geqslant 1),$$

it follows that

$$\alpha_j^F(R)\leqslant\alpha_j(R)<1.$$

Therefore Theorem 2.1 applies to $F=\mathscr{B}_\gamma f$, and thus $\mathscr{B}_\gamma f$ is of bounded l-index in $\mathbb{D}_R$ for $l(r)=1/(R-r)$.

Finally, $\alpha_j^F(R)\leqslant\alpha_j(R)$ and the function $x\mapsto(1+x)/(1-x)$ is increasing on $(0,1)$, the same integer $J(R)$ is admissible for F. Hence

$$N(\mathscr{B}_\gamma f,l;\mathbb{D}_R)\leqslant j+J(R).$$

□

4. Ruscheweyh derivative

For $s \in \mathbb{N}$, the Ruscheweyh derivative is defined by

$$R^s f(z) = \frac{z}{s!}\bigl(z^{s-1} f(z)\bigr)^{(s)},$$

and, in the coefficient form,

$$(R^s f)(z) = \sum_{m=j}^{\infty} c_m z^m, \qquad c_m = \frac{(m)_s}{s!} a_m,$$

where

$$(m)_s = m(m+1)\cdots(m+s-1);$$

see [1].

Proposition 4.1. *Let*

$$f(z) = \sum_{m=j}^{\infty} a_m z^m, \qquad a_j \neq 0, \qquad j \geqslant 1,$$

be analytic in $\mathbb{D}_R$, *and let* $s \in \mathbb{N}$. *Assume that*

$$\beta_{j,s}(R) := \sum_{n=1}^{\infty} \frac{(n+j)!}{j!\,n!} \frac{(n+j)_s}{(j)_s} \frac{|a_{n+j}|}{|a_j|} R^n < 1.$$

Then $R^s f$ *is of bounded* l*-index in* $\mathbb{D}_R$ *for*

$$l(r) = \frac{1}{R-r}.$$

More precisely,

$$N(R^s f, l; \mathbb{D}_R) \leqslant j + J_s(R),$$

where

$$J_s(R) := \min\left\{ k \in \mathbb{Z}_+ : \ k \geqslant j, \ \frac{1+\beta_{j,s}(R)}{1-\beta_{j,s}(R)} \leqslant \frac{(k+j)!}{k!\,j!} \right\}.$$

Proof. Put $G := R^s f$. Since $(j)_s > 0$, the first nonzero coefficient of G is again the coefficient of z^j. Therefore

$$\alpha_j^G(R) = \sum_{n=1}^{\infty} \frac{(n+j)!}{j!\,n!} \frac{|c_{n+j}|}{|c_j|} R^n.$$

Using the coefficient formula for c_m, we obtain

$$\frac{|c_{n+j}|}{|c_j|} = \frac{(n+j)_s}{(j)_s}\frac{|a_{n+j}|}{|a_j|}.$$

Hence,

$$\alpha_j^G(R) = \sum_{n=1}^{\infty} \frac{(n+j)!}{j!n!}\frac{(n+j)_s}{(j)_s}\frac{|a_{n+j}|}{|a_j|}R^n = \beta_{j,s}(R).$$

By assumption, $\beta_{j,s}(R) < 1$, so Theorem 2.1 applies directly to $G = R^s f$. Therefore, $R^s f$ is of bounded l-index in $\mathbb{D}_R$ for $l(r) = 1/(R-r)$, and

$$N(R^s f, l; \mathbb{D}_R) \leqslant j + J_s(R).$$

□

Remark 4.1. Note that in essence, the established statements in the case of the Bernardi operator concern only the subclass of univalent functions, which is described by the sufficient condition of the theorem from [7] . In the case of the Rusheweyh derivative, we conside an even narrower class of univalent functions. In general, the problem formulated by Prof. O.B. Skaskiv remains, in general, open.

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ПРО ЗБЕРЕЖЕННЯ ОБМЕЖЕНОСТІ *L*-ІНДЕКСУ ПІД ДІЄЮ ІНТЕГРАЛЬНОГО ОПЕРАТОРА БЕРНАРДІ ТА ПОХІДНОЇ РУШЕВЕЯ

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Досліджується дія інтегрального оператора Бернарді і оператора похідної Рушевея на аналітичні функції обмеженого l-індексу в крузі. Вказано достатні умови, яким повинні задовольняти функції обмеженого l-індексу, за яких їхні образи під дією інтегрального оператора Бернарді, чи під дією оператора похідної Рушевея, будуть функціями обмеженого l-індексу з тією ж функцією l. В основу доведення покладено теорему М.М. Шеремети і З.М. Шеремети (1999), яка містить умови на коефіцієнти Тейлора аналітичної функції, достатні для обмеженості l-індексу даної аналітичної функції в крузі.

Ключові слова: *однолиста функція; обмежений l-індекс; інтегральний оператор Бернарді; похідна Рушевея.*