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SUBSYMMETRIC LINEAR CONTINUOUS FUNCTIONALS ON SPACES OF LEBESGUE INTEGRABLE FUNCTIONS ON THE SEMI-AXIS

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The work is devoted to the study of subsymmetric continuous linear functionals on real and complex topological vector spaces of Lebesgue integrable functions on the semi-axis. We consider the class of such spaces satisfying some natural conditions. The complete description of the structure of a subsymmetric continuous linear functional defined on an arbitrary representative of this class has been found. Specifically, we show that every such a functional can be represented as an integral of its argument up to multiplication by a constant.

Key words: *linear functional, subsymmetric function, Banach space, Lebesgue integrable function.*

1. Introduction

Subsymmetric polynomials and, in particular, subsymmetric linear functionals on sequence Banach spaces were studied in [1] and [2]. In this work, we consider the notion of subsymmetry for functions defined on some topological vector spaces of Lebesgue integrable functions on the semi-axis. We describe the structure of subsymmetric continuous linear functionals on these spaces. Namely, we show that every such a functional can be represented as the Lebesgue integral over the semi-axis multiplied by some constant number. Result of this work can be used in the investigations of subsymmetric polynomials on spaces of Lebesgue integrable functions.

2. Results

For $A \subset [0,+\infty)$ and for $t \in [0,+\infty)$ we define

$$1_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For $n \in \mathbb{N}$, let D_n be the set of all functions $x : [0,+\infty) \to \mathbb{K}$ such that, for every $j \in \mathbb{N}$,

$$x(t) = a_j \tag{1}$$

for $t \in \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right)$, where $a_1, a_2, \ldots \in \mathbb{K}$ are such that $\sum_{j=1}^{\infty} |a_j| < \infty$. Denote $D = \bigcup_{n=1}^{\infty} D_n$. Let $X[0,+\infty)$ be a topological vector space over $\mathbb{K}$ of classes of equivalence (with respect to the equivalence relation $x_1 \sim x_2 \iff x_1 \overset{\text{a.e.}}{=} x_2$) of Lebesgue measurable functions $x : [0,+\infty) \to \mathbb{K}$ with the usual operations of addition and multiplication by scalars such that the following conditions are satisfied:

(1) Every $x \in X[0,+\infty)$ is Lebesgue integrable, i.e. the integral

$$\int_{[0,+\infty)} |x(t)|\, dt$$

is finite.

(2) The functional $g : X[0,+\infty) \to \mathbb{K}$, defined by

$$g(x) = \int_{[0,+\infty)} x(t)\, dt, \tag{2}$$

where $x \in X[0,+\infty)$, is continuous.

(3) The set D is dense in $X[0,+\infty)$.

(4) For every $n \in \mathbb{N}$ and for every $x \in D_n$ of the form (1), the series $\sum_{j=1}^{\infty} a_j 1_{\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right]}$ is weakly convergent to x.

(5) For every $a, b \in [0,+\infty)$ such that $a < b$ and for every $x \in X[0,+\infty)$, the function $\beta_{[a,b]}(x) : [0,+\infty) \to \mathbb{K}$, defined by

$$\beta_{[a,b]}(x)(t) = \begin{cases} x(t), & \text{if } t < a, \\ 0, & \text{if } t \in [a,b], \\ x(t-b+a), & \text{if } t > b \end{cases} \tag{3}$$

for $t \in [0,+\infty)$, belongs to $X[0,+\infty)$.

In particular, the space $L_1[0,+\infty)$ of all $\mathbb{K}$-valued Lebesgue integrable functions on $[0,+\infty)$ can serve as an instance of $X[0,+\infty)$.

A function $f : X[0,+\infty) \to \mathbb{K}$ is called subsymmetric if

$$f\big(\beta_{[a,b]}(x)\big) = f(x)$$

for every $x \in X[0,+\infty)$ and $a,b \in [0,+\infty)$ such that $a < b$, where $\beta_{[a,b]}$ is defined by (3).

Our goal is to describe all subsymmetric linear continuous functionals on $X[0,+\infty)$. First, we prove some auxiliary results.

Lemma 2.1. *Let $f : X[0,+\infty) \to \mathbb{K}$ be a subsymmetric function. Then, for every $a,b,c \in [0,+\infty)$ such that $a < b$,*

$$f\big(1_{[a,b]}\big) = f\big(1_{[a+c,b+c]}\big).$$

Proof. By (3),

$$\beta_{[0,c]}\big(1_{[a,b]}\big) = 1_{[a+c,b+c]}.$$

Therefore, since f is subsymmetric,

$$f\big(1_{[a,b]}\big) = f\big(1_{[a+c,b+c]}\big).$$

This completes the proof. □

Lemma 2.2. *Let $f : X[0,+\infty) \to \mathbb{K}$ be a subsymmetric linear functional. Then, for every $a \in [0,+\infty)$ and $m \in \mathbb{N}$,*

$$f\big(1_{[0,ma]}\big) = mf\big(1_{[0,a]}\big).$$

Proof. Since

$$[0,ma] = [0,a] \sqcup (a,2a] \sqcup \ldots \sqcup ((m-1)a,ma],$$

it follows that

$$1_{[0,ma]} = 1_{[0,a]} + 1_{(a,2a]} + \ldots + 1_{((m-1)a,ma]}.$$

Therefore, by the linearity of f,

$$f\big(1_{[0,ma]}\big) = f\big(1_{[0,a]}\big) + f\big(1_{(a,2a]}\big) + \ldots + f\big(1_{((m-1)a,ma]}\big). \tag{4}$$

Since $1_{(a,2a]} \overset{\text{a.e.}}{=} 1_{[a,2a]}, \ldots, 1_{((m-1)a,ma]} \overset{\text{a.e.}}{=} 1_{[(m-1)a,ma]}$, it follows that

$$f\big(1_{(a,2a]}\big) = f\big(1_{[a,2a]}\big), \ldots, f\big(1_{((m-1)a,ma]}\big) = f\big(1_{[(m-1)a,ma]}\big)$$

and, consequently, by (4),

$$f\left(1_{[0,ma]}\right) = f\left(1_{[0,a]}\right) + f\left(1_{[a,2a]}\right) + \ldots + f\left(1_{[(m-1)a,ma]}\right). \tag{5}$$

By Lemma 2.1, one has

$$f\left(1_{[0,a]}\right) = f\left(1_{[a,2a]}\right) = \ldots = f\left(1_{[(m-1)a,ma]}\right).$$

Therefore, by (5), we obtain

$$f\left(1_{[0,ma]}\right) = mf\left(1_{[0,a]}\right).$$

This completes the proof. □

Lemma 2.3. *Let $f : X[0,+\infty) \to \mathbb{K}$ be a subsymmetric linear functional. Then, for every $n, j_1, j_2 \in \mathbb{N}$ such that $j_1 < j_2$,*

$$f\left(1_{\left[\frac{j_1}{2^n},\frac{j_2}{2^n}\right]}\right) = \mu\left(\left[\frac{j_1}{2^n},\frac{j_2}{2^n}\right]\right) f\left(1_{[0,1]}\right),$$

where μ is the Lebesgue measure on $[0,+\infty)$.

Proof. Let $n, j_1, j_2 \in \mathbb{N}$ be such that $j_1 < j_2$, Note that

$$\left[0,\frac{j_2}{2^n}\right] = \left[0,\frac{j_1}{2^n}\right] \sqcup \left(\frac{j_1}{2^n},\frac{j_2}{2^n}\right].$$

Therefore

$$1_{\left[0,\frac{j_2}{2^n}\right]} = 1_{\left[0,\frac{j_1}{2^n}\right]} + 1_{\left(\frac{j_1}{2^n},\frac{j_2}{2^n}\right]}.$$

Consequently, by the linearity of f,

$$f\left(1_{\left[0,\frac{j_2}{2^n}\right]}\right) = f\left(1_{\left[0,\frac{j_1}{2^n}\right]}\right) + f\left(1_{\left(\frac{j_1}{2^n},\frac{j_2}{2^n}\right]}\right). \tag{6}$$

Since $1_{\left(\frac{j_1}{2^n},\frac{j_2}{2^n}\right]} \overset{\text{a.e.}}{=} 1_{\left[\frac{j_1}{2^n},\frac{j_2}{2^n}\right]}$, it follows that

$$f\left(1_{\left(\frac{j_1}{2^n},\frac{j_2}{2^n}\right]}\right) = f\left(1_{\left[\frac{j_1}{2^n},\frac{j_2}{2^n}\right]}\right).$$

Consequently, by (6), the following equality is valid

$$f\left(1_{\left[0,\frac{j_2}{2^n}\right]}\right) = f\left(1_{\left[0,\frac{j_1}{2^n}\right]}\right) + f\left(1_{\left[\frac{j_1}{2^n},\frac{j_2}{2^n}\right]}\right).$$

Therefore,

$$f\left(1_{\left[\frac{j_1}{2^n},\frac{j_2}{2^n}\right]}\right)=f\left(1_{\left[0,\frac{j_2}{2^n}\right]}\right)-f\left(1_{\left[0,\frac{j_1}{2^n}\right]}\right). \tag{7}$$

By Lemma 2.2, we deduce

$$f\left(1_{\left[0,\frac{j_1}{2^n}\right]}\right)=j_1 f\left(1_{\left[0,\frac{1}{2^n}\right]}\right) \quad \text{and} \quad f\left(1_{\left[0,\frac{j_2}{2^n}\right]}\right)=j_2 f\left(1_{\left[0,\frac{1}{2^n}\right]}\right).$$

Therefore, by (7),

$$f\left(1_{\left[\frac{j_1}{2^n},\frac{j_2}{2^n}\right]}\right)=(j_2-j_1)f\left(1_{\left[0,\frac{1}{2^n}\right]}\right). \tag{8}$$

By Lemma 2.2, one has

$$f\left(1_{[0,1]}\right)=2^n f\left(1_{\left[0,\frac{1}{2^n}\right]}\right).$$

Therefore, by (8),

$$f\left(1_{\left[\frac{j_1}{2^n},\frac{j_2}{2^n}\right]}\right)=\left(\frac{j_2}{2^n}-\frac{j_1}{2^n}\right)f\left(1_{[0,1]}\right)$$

and, since $\mu\left(\left[\frac{j_1}{2^n},\frac{j_2}{2^n}\right]\right)=\frac{j_2}{2^n}-\frac{j_1}{2^n}$, it follows that

$$f\left(1_{\left[\frac{j_1}{2^n},\frac{j_2}{2^n}\right]}\right)=\mu\left(\left[\frac{j_1}{2^n},\frac{j_2}{2^n}\right]\right)f\left(1_{[0,1]}\right).$$

This completes the proof. □

Theorem 2.1. *Let $f : X[0,+\infty) \to \mathbb{K}$ be a subsymmetric linear continuous functional. Then*

$$f(x)=k\int_{[0,+\infty)} x(t)\,dt \tag{9}$$

for every $x \in X[0,+\infty)$, where $k = f\left(1_{[0,1]}\right)$.

Proof. First, let us show that (9) holds for every $x \in D$. Let $x \in D$. Then x has the form (1) for some $n \in \mathbb{N}$. Taking into account the continuity and the linearity of f, and Condition 4 imposed on the space $X[0,+\infty)$, we can write

$$f(x)=\sum_{j=1}^{\infty} a_j f\left(1_{\left[\frac{j-1}{2^n},\frac{j}{2^n}\right]}\right). \tag{10}$$

By Lemma 2.3,

$$f\left(1_{\left[\frac{j-1}{2^n},\frac{j}{2^n}\right]}\right)=\mu\left(\left[\frac{j-1}{2^n},\frac{j}{2^n}\right]\right)f\left(1_{[0,1]}\right)$$

for every $j\in\mathbb{N}$. Consequently, by (10),

$$f(x)=f\left(1_{[0,1]}\right)\sum_{j=1}^{\infty}a_j\mu\left(\left[\frac{j-1}{2^n},\frac{j}{2^n}\right]\right). \tag{11}$$

Note that

$$\int_{[0,+\infty)}x(t)\,dt=\sum_{j=1}^{\infty}a_j\mu\left(\left[\frac{j-1}{2^n},\frac{j}{2^n}\right]\right).$$

Therefore, by (11), the equality (9) holds. So, (9) holds for every $x\in D$. Therefore $f(x)=kg(x)$ for every $x\in D$, where the functional g is defined by (2) and $k=f\left(1_{[0,1]}\right)$. By Condition 2 on $X[0,+\infty)$, the functional g is continuous. By Condition 3 on $X[0,+\infty)$, the set D is dense in $X[0,+\infty)$. So, the linear continuous functionals f and kg coincide on the dense subset of $X[0,+\infty)$. Consequently, $f(x)=kg(x)$ for every $x\in X[0,+\infty)$. This completes the proof. □

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СУБСИМЕТРИЧНІ ЛІНІЙНІ НЕПЕРЕРВНІ ФУНКЦІОНАЛИ НА ПРОСТОРАХ ІНТЕГРОВНИХ ЗА ЛЕБЕГОМ ФУНКЦІЙ НА ПІВОСІ

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Дану роботу присвячено дослідженню субсиметричних неперервних лінійних функціоналів на дійсних і комплексних топологічних векторних просторах інтегровних за Лебегом функцій на півосі. Виділено клас таких просторів, які задовольняють певним природним умовам. Зроблено повний опис структури субсиметричних неперервних лінійних функціоналів, визначених на довільному представнику цього класу. А саме, показано, що кожен такий функціонал може бути поданий у вигляді інтеграла від свого аргументу з точністю до сталого множника.

Ключові слова: *лінійний функціонал, субсиметрична функція, банахів простір, інтегровна за Лебегом функція.*