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IDENTIFICATION OF FUNCTIONS IN HARDY SPACE ON A HALF-STRIP DOMAIN

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This paper explores a signal identification problem using a convolution. We identify the signal with some function belonging to the Hardy space in a domain outside the half-strip. For this, we use a test function belonging to the Hardy space in the half-strip. Signals generated by point singularities and singularities generated by a segment are considered. The exponential test functions are chosen. For some cases, the exact solutions are found, for others, a convolution is found, and the possibility of finding the desired signal is analyzed.

Key words: *Hardy space, identification of signals, convolution equation.*

1. Introduction

The signal detection is a fundamental problem in many scientific and engineering fields, as it enables the identification of meaningful information within noisy data. In communications, accurate signal detection ensures reliable data transmission over imperfect channels. In radar, sonar, and medical imaging, it allows the recognition of objects or physiological patterns that may otherwise be hidden by interference. Effective signal detection directly impacts the performance, accuracy, and safety of modern systems — from wireless networks and navigation to diagnostics and defense. Therefore, developing robust methods for signal detection is crucial for improving information processing and decision-making in complex environments. One of the main methods of signal identification theory is the Wiener filter method for L^2-spaces. The classic view of the problem is contained in [1], [5].

We will consider the problem of identifying a signal in the half-strip domain of the complex plane.

Let $D_\sigma = \{z : |\mathrm{Im} z| < \sigma, \mathrm{Re} z < 0\}$ and $D^*_\sigma = \mathbb{C}\backslash\overline{D}_\sigma$. Suppose $E^p[D_\sigma]$ and $E^p[D^*_\sigma], 1 \le p < +\infty, \sigma > 0$, are the spaces of analytic functions respectively in the domains D_σ and D^*_σ, for which

$$\|f\| := \sup\left\{\int\limits_\gamma |f(z)|^p|dz|\right\}^{1/p} < +\infty,$$

where supremum is taken over all segments γ, that lay in D_σ and D^*_σ, respectively.

We identify the signal with the function $g \in E^2[D^*_\sigma]$ as in [2].

B. Vynnytskyi and V. Dilnyi [3, 4] described all functions $g \in E^2_*[D_\sigma]$, for which the convolution

$$T(\tau) = \int\limits_{\partial D_\sigma} f(w+\tau)g(w)dw, \tau \le 0, \tag{1.1}$$

is the identical zero for all $f \in E^2[D_\sigma]$. The main idea in these studies was to use the amplitude spectrum

$$G(z) = \frac{1}{i\sqrt{2\pi}}\int\limits_{\partial D_\sigma} g(w)e^{zw}dw.$$

The aim of this paper is to identify the functions g generated by one-point singularity in D_σ, that is $g(w) = \frac{1}{w-w_0}$, $w_0 \in D_\sigma$, using some test function f. The problem of identifying of g is identical to finding the point w_0.

In [3], [4] it is proven that for $g_0(w) = \frac{1}{w-w_0}$, $w_0 \in D_\sigma$, its amplitude spectrum is cyclic. Hence, no such function $f \in E^2[D_\sigma]$ exists that $T(\tau) \equiv 0$, $\tau \le 0$. Therefore, every function $f \in E^2[D_\sigma]$ can be used to obtain some information about the $g_0(w) = \frac{1}{w-w_0}$ i.e. $w_0 \in D_\sigma$. But not every such function f provides enough information to identify the function g_0.

2. Identification of functions by convolution

Consider three types of unknown signals.

1) Let $g(w) = \frac{1}{w-w_0}$, $w_0 \in D_\sigma$. This function has a *single* point in $\mathbb{C}$, where it is not analytic. We choose filter functions $f(w) = e^{ww_1}$, $\mathrm{Re}(w_1) \in \mathbb{C}_+$. Then

$$T(\tau) = \int\limits_{\partial D_\sigma} e^{(w_1(w+\tau))}\frac{1}{w-w_0}dw = e^{w_1\tau}\int\limits_{\partial D_\sigma}\frac{e^{w_1 w}}{w-w_0}dw.$$

Since $\frac{e^{w_1 w}}{w-w_0}$ has the simple pole in $w_0 \in D_\sigma$, we have

$$T(\tau) = e^{w_1\tau} 2\pi i \operatorname{Res}_{w=w_0} \frac{e^{w_1 w}}{w-w_0} dw = e^{w_1\tau} 2\pi i e^{w_1 w_0}.$$

Hence,

$$w_0 = \frac{\ln(T(\tau)/2\pi i)}{w_1} - \tau, \tag{2.1}$$

where the branch of the logarithm is chosen so that the point w_0 is contained in D_σ.

So we can formulate the following elementary statement.

Proposition 2.1. *Let $g(w) = \frac{1}{w-w_0}$ with an unknown parameter $w_0 \in D_\sigma$. Then w_0 can be defined by equality (2.1), where convolution T is defined by (1.1).*

2) Let now $g(w) = \sum\limits_{k=1}^{n} \frac{a_k}{w-\widetilde{w}_k}$, $\widetilde{w}_k \in D_\sigma$. This function is non-analytic only in the points $\widetilde{w}_1$, $\widetilde{w}_2$, ... $\widetilde{w}_n$. Then for $f(w) = e^{ww_1}$, $\operatorname{Re}(w_1) \in \mathbb{C}_+$, we have

$$T(\tau) = \int\limits_{\partial D_\sigma} \sum_{k=1}^{n} e^{(w_1(w+\tau))} \frac{a_k}{w-\widetilde{w}_k} dw = e^{w_1\tau} \int\limits_{\partial D_\sigma} \sum_{k=1}^{n} \frac{a_k e^{w_1 w}}{w-\widetilde{w}_k} dw.$$

Since $\frac{e^{w_1 w}}{w-\widetilde{w}_k}$ has the simple pole in $\widetilde{w}_k \in D_\sigma$, we have

$$T(\tau) = e^{w_1\tau} 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{w=\widetilde{w}_k} \frac{a_k e^{w_1 w}}{w-\widetilde{w}_k} dw = e^{w_1\tau} 2\pi i \sum_{k=1}^{n} a_k e^{w_1 \widetilde{w}_k}.$$

Choosing a set of values $w_1 \in \mathscr{W} = \{w_{11}, w_{12}, \ldots, w_{1n}\}$ for the test function f and analyzing the fraction

$$\frac{T(\tau)}{2\pi i e^{w_1\tau}},$$

one can numerically calculate $\widetilde{w}_1$, $\widetilde{w}_2$, ... , $\widetilde{w}_n$ as the solution of some system of exponential equations (or algebraic equations concerning $e^{w_1 \widetilde{w}_k}$).

3) Suppose $g(w) = \int\limits_{\widetilde{w}_0}^{\widetilde{w}_1} \frac{1}{z-w} dz$, where $\widetilde{w}_0$, $\widetilde{w}_1 \in D_\sigma$, and integration is over segment $[\widetilde{w}_0; \widetilde{w}_1] \subset D_\sigma$. This function g is a Cauchy-type integral and therefore g is analytic in $\mathbb{C}$ outside the segment $[\widetilde{w}_0; \widetilde{w}_1]$. We interpret this fact in such a way that the singularity of g is generated by this segment.

For $f(w) = e^{ww_1}$ we obtain

$$T(\tau) = \int\limits_{\partial D_\sigma} e^{(w_1(w+\tau))} \int\limits_{\widetilde{w}_0}^{\widetilde{w}_1} \frac{1}{z-w} dz dw =$$

$$= \int\limits_{\partial D_\sigma} e^{(w_1(w+\tau))} (\ln(w-\widetilde{w}_1) - \ln(w-\widetilde{w}_0)) dw,$$

where $\ln(w-\mu)$ is the main branch of logarithm with the cut $\{w = u + i\mathrm{Im}\mu : u \in (-\infty; \mathrm{Re}\mu)\}$. Therefore

$$T(\tau) = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_0^{-\infty} e^{(\tau+u+i\sigma)w_1} \left(\log(u+i\sigma-\widetilde{w_1}) - \log(u+i\sigma-\widetilde{w_0})\right) du =$$

$$= -\frac{e^{w_1\tau+iw_1\sigma}}{w_1} \left(\log(i\sigma-\widetilde{w_1}) - \log(i\sigma-\widetilde{w_0})\right) -$$

$$-\frac{1}{w_1}\left(e^{w_1\tau+w_1\widetilde{w_1}}\,\mathrm{Ei}\big(w_1(i\sigma-\widetilde{w_1})\big) - e^{w_1\tau+w_1\widetilde{w_0}}\,\mathrm{Ei}\big(w_1(i\sigma-\widetilde{w_0})\big)\right),$$

$$I_2 = \int_{-\infty}^0 e^{(\tau+u-i\sigma)w_1} \left(\log(u-i\sigma-\widetilde{w_1}) - \log(u-i\sigma-\widetilde{w_0})\right) du =$$

$$= \frac{e^{w_1\tau-iw_1\sigma}}{w_1} \left(\log(-i\sigma-\widetilde{w_1}) - \log(-i\sigma-\widetilde{w_0})\right) -$$

$$-\frac{1}{w_1}\left(e^{w_1\tau+w_1\widetilde{w_1}}\,\mathrm{Ei}\big(w_1(-i\sigma-\widetilde{w_1})\big) - e^{w_1\tau+w_1\widetilde{w_0}}\,\mathrm{Ei}\big(w_1(-i\sigma-\widetilde{w_0})\big)\right),$$

$$I_3 = \int_{-\sigma}^{\sigma} e^{(\tau+iy)w_1} \left(\log(iy-\widetilde{w_1}) - \log(iy-\widetilde{w_0})\right) dy =$$

$$= e^{w_1\tau}\Bigg[\frac{1}{iw_1}\Big(e^{iw_1\sigma}\big(\log(i\sigma-\widetilde{w_1}) - \log(i\sigma-\widetilde{w_0})\big) -$$

$$e^{-iw_1\sigma}\big(\log(-i\sigma-\widetilde{w_1}) - \log(-i\sigma-\widetilde{w_0})\big)\Big) -$$

$$-\frac{1}{iw_1}\Big(e^{w_1\widetilde{w_1}}\big(\mathrm{Ei}(w_1(i\sigma-\widetilde{w_1})) - \mathrm{Ei}(w_1(-i\sigma-\widetilde{w_1}))\big) -$$

$$-e^{w_1\widetilde{w_0}}\Big(\mathrm{Ei}(w_1(i\sigma-\widetilde{w_0}))-\mathrm{Ei}(w_1(-i\sigma-\widetilde{w_0}))\Big)\Bigg].$$

Then after the notification

$$H(z):=(1+i)\Big(-e^{iw_1\sigma}\log(i\sigma-z)+e^{-iw_1\sigma}\log(-i\sigma-z)+$$
$$+e^{w_1 z}\Big(\mathrm{Ei}\big(w_1(i\sigma-z)\big)-\mathrm{Ei}\big(w_1(-i\sigma-z)\big)\Big)$$

we have

$$T(\tau)=\frac{e^{w_1\tau}}{w_1}\big(H(\widetilde{w}_1)-H(\widetilde{w}_0)\big), \tag{2.2}$$

where branch cut of Ei is taken on the positive real axis. At the fact, we proved the following simple statement

Proposition 2.2. *For a signal defined by the segment-generated singularity* $g(w)=\int\limits_{\widetilde{w}_0}^{\widetilde{w}_1}\frac{1}{z-w}dz$ *with* $\widetilde{w}_0$, $\widetilde{w}_1\in D_\sigma$ *and a test function* $f(w)=e^{ww_1}$ *its convolution is given by* (2.2).

Finding the unknown parameters $\widetilde{w}_1$ and $\widetilde{w}_0$ in Equation (2.2) is quite difficult in this case. This shows that a test function that is successfully used to identify one type of signal may be less useful for another type.

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ІДЕНТИФІКАЦІЯ ФУНКЦІЙ У ПРОСТОРІ ГАРДІ У ПІВСМУЗІ

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У статті досліджується задача ідентифікації сигналів за допомогою згортки. Сигнал ми ототожнюємо з деякою функцією, що належить простору Гарді в області поза півсмугою. Для цього використовуємо тестову функцію, що належить простору Гарді у півсмузі. Розглядаються сигнали, породжені точковими сингулярностями, а також сингулярностями, породженими відрізком. Обрано експоненційні тестові функції. Для деяких випадків знайдено точні розв'язки, для інших знайдено згортку і проаналізовано можливість знаходження шуканого сигналу.

Ключові слова: *простір Гарді, ідентифікація сигналів, рівняння згортки.*