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IMAGE PROCESSING BY KRAVCHUK POLYNOMIALS: OPEN QUESTION

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The study focuses on constructing spectral models of functions based on classical orthogonal polynomials, particularly Kravchuk polynomials, which serve as discrete analogs of continuous orthogonal bases. The use of such bases increases computational stability and speed, reduces approximation errors, and ensures the analytical representation of diagnostic signals in digital form. Discrete-variable bases are shown to be the most suitable for the development of software in non-destructive testing and diagnostic systems, as they eliminate the need for complex numerical integration. Special attention is paid to the analytical properties of Kravchuk polynomials, their recurrence relations, and their potential application in spectral transformations of functions. The problem of applying Kravchuk polynomials of several variables for image/speeach processing is formulated.

Key words: *bivariate Kravchuk polynomial; orthogonal polynomial; Krawtchouk polynomial.*

1. Introduction

It is fundamentally possible to perform analytical function transformations using only the spectral representations of functions. Moreover, this approach yields results that are often more practically advantageous compared to those obtained via numerical methods. This is explained by the high adaptability of the method when choosing a basis from families of classical orthogonal polynomials, the smoothness of the approximated representation defined by a finite orthogonal series, and other factors.

The primary drawback of spectral transformations of functions is that the computational process is often excessively complex and cumbersome. Such algorithms require stages of preliminary computation and the reservation of large memory volumes for auxiliary coefficients. These limitations pose significant obstacles for implementing this approach in applications requiring fast, stable, and accurate analytical transformations of functions and signals. The time overhead may force software developers to favor coarser but faster algorithms.

This situation underscores the need to develop an alternative, simple, and precise mathematical framework for rapid spectral transformations of functions that correspond to a range of essential analytical function transformations and their superpositions. As orthogonal bases, all systems of classical orthogonal polynomials (Laguerre, Jacobi, Hermite, Legendre, Sonine-Laguerre, Gegenbauer, Chebyshev polynomials of the first and second kinds) and their principal modifications are considered.

The solution of the problem of creating a mathematical apparatus for rapid analytical transformations of functions represented by finite orthogonal series constitutes a cornerstone of modern computer science development. Functional systems constructed on the basis of classical orthogonal polynomials serve as the orthogonal bases.

The family of Kravchuk polynomials was first introduced in 1929 in connection with the binomial distribution in probability theory. These polynomials form an orthogonal system on a discrete set and play a crucial role in the discrete analogue of the Fourier transform—known as the Kravchuk Transform—which is widely applied in digital image processing, data compression, biometric identification, and computer vision [1–4].

For example, in face recognition tasks, the Kravchuk Transform is employed to extract invariant image features that remain robust under rotation and scaling transformations [5]. In other domains, such as quantum computation, the orthogonal Kravchuk polynomials enable the construction of exact solutions to the Schrödinger equation for certain classes of potentials, as well as the description of quantum oscillator spectra.

2. Univariate Kravchuk Polynomials

Denoted as $K_n^{p,q}(x)$, they are given on the interval $[0,N]$, where constants $p>0,\ q>0,\ p+q=1$. They are represented by the Rodrigue formula (discrete analogue):

$$K_n^{p,q}(x)=\frac{(-1)^n x!\,(N-x)!}{n!\,p^x q^{N-x}} \tag{1}$$

At the same time

$$\|K_n^{p,q}(x)\|=\sqrt{\frac{N!}{n!\,(N-n)!}}\cdot pq^{\frac{n}{2}} \tag{2}$$

Recurrence ratio

$$(n-1)\,K_{n+1}^{p,q}(x)=[x-n-p\,(N-2n)]\,K_n^{p,q}(x)-pq\,(N-n+1)\,K_{n-1}^{p,q}(x). \quad (3)$$

Jump function is

$$j\,(x)=\frac{N!}{n!\,(N-n)!}p^x q^{N-x} \quad (4)$$

The eigenvalues are:

$$K_0^{p,q}(x)=1$$
$$K_1^{p,q}(x)=(p+q)x-Np$$
$$K_2^{p,q}(x)=\frac{p^2}{2}(N-x)(N-x-1)-pq(N-x)x+\frac{q^2}{2}x(x-1)$$
$$K_3^{p,q}(x)=\tfrac{p^3}{6}(N-x)(N-x-1)(N-x-2)+\tfrac{p^2q}{2}x(N-x)(N-x-1)-$$
$$-\tfrac{q^2p}{2}x(x-1)(N-x)+\tfrac{q^3}{6}(x-1)(x-2)x$$

Formula for calculating coefficients of the Kravchuk polynomial decomposition of the function $f\,(x)$ has the form:

$$B_n=\sum_{x=0}^{N} f\,(x)\,K_n^{p,q}(x)\,\frac{N!}{(N-x)!x!}p^x q^{N-x}, \quad (5)$$

where $K_n^{p,q}(x)$ is Kravchuk orthonormal polynomials.

Function recovery formula $f\,(x)$

$$f\,(x)=\sum_{n=0}^{N} B_n K_n^{p,q}(x). \quad (6)$$

Kravchuk functions denote $k_n^{p,q}(x)$ they are obtained similarly to the Meixner functions, i.e.

$$k_n^{p,q}(x)=K_n^{p,q}(x)\sqrt{\frac{N!}{(N-x)!x!}}p^{0,5x}q^{0,5(N-x)}. \quad (7)$$

Taking into account (5) – (7), the decomposition of the Kravchuk functions $f\,(x)$ requires the calculation of the decomposition coefficients according to the formula

$$B_n=\sum_{x=0}^{N} f\,(x)\,k_n^{p,q}(x). \quad (8)$$

In this case, the recovery $f(x)$ should be performed according to the formula

$$f(x) = \sum_{n=0}^{N} B_n k_n^{p,q}(x) = \sum_{n=0}^{N} B_n K_n^{p,q}(x) \sqrt{\frac{N!}{(N-x)!x!}} p^{0,5x} q^{0,5(N-x)}. \quad (9)$$

3. Open question

Main idea of digital image and signal processing is the following (see [1]): as an efficient data descriptor is used some moments set. It can be used to represent data without information redundancy in the moment set and is able to detect any small variation in data intensity [6]. Moments for a one-dimensional signal $f(x)$ with length N can be defined as

$$Q_n = \sum_{x=0}^{n-1} R_n(x) f(x),\ n \in \{0,1,2,\ldots,N-1\}.$$

Here $R_n(x)$ is some orthogonal polynomial. In particular, $R_n(x)$ can equal with the Kravchuk polynomial $K_n^{p,1-p}(x)$ The signal can be reconstructed from the moments using the inverse moment transformations

$$f(x) = \sum_{n=0}^{N-1} Q_n R_n(x),\quad x \in \{0,1,2,\ldots,N-1\}.$$

In the case of a two-dimensional signal $f(x,y)$ with size $N \times N$, the discrete proposed transform is given by

$$Q_{nm} = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} R_n(x) R_m(y) f(x,y),$$

with $n,m \in \{0,1,2,\ldots,N-1\}$, and the discrete inverse transform is

$$f(x,y) = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} Q_{nm}(x) R_n(x) R_m(y),$$

$x,\ y \in \{0,1,2,\ldots,N-1\}$.

But such an approach uses one-dimensional Kravchuk (or orther othogonal) polynomials. An two-dimensional aspects are considered in the transformation as separable product of $R_n(x)R_m(y)$. Since the signal function $f(x,y)$ is arbitrary bivariate function, the representation may cause loss of a significant portion of the image or signal by this separate two-dimnensionality. It leads to such a question:

Problem 1. *Is it possible in image/speech processing tasks with lesser loss of quality to replace the product $R_n(x)R_m(y)$ by bivariate Kravchuk polynomials, introduced in [7–11]?*

4. Discussion

The fundamental analytical relations that define the properties of various orthogonal bases of a discrete variable, belonging to the class of classical systems, represent discrete analogues of the corresponding formulas that determine the properties of classical orthogonal bases of a continuous variable. This indicates that many characteristics of orthogonal systems of continuous and discrete variables are, in fact, nearly identical. The use of orthogonal bases of a discrete variable is primarily motivated by the need to avoid numerical integration of complex functional dependencies, replacing it with summation operations when computing decomposition coefficients on digital computers.

Discrete-variable bases are particularly well-suited for the development of software systems – for example, diagnostic instruments, scanners, and other digital devices – while remaining free from the drawbacks inherent to discrete orthogonal systems constructed from piecewise-constant, non-differentiable functions (such as Haar, Walsh, or Hadamard functions). Although these latter systems are easily implementable on computers, they do not provide analytical representations of informational data, thereby excluding analytical procedures from the processing chain entirely.

In contrast, analytical approximation using discrete orthogonal bases enables the implementation of adaptive procedures (analogous to those applied with continuous bases), which significantly enhance the efficiency of data representation. This allows one to achieve a prescribed level of analytical accuracy for input data arrays using the shortest possible segment of the orthogonal series.

Consequently, the continuous growth of information volume and the increasing demand for precision and completeness of data processing within minimal time frames compel developers of modern information systems to increase their computational capacities. However, the ongoing escalation of processing speed often leads to higher operational costs and does not necessarily guarantee compliance with essential data-processing requirements. Therefore, there is an urgent mathematical challenge to develop new methods of data processing that can ensure the required accuracy and computational speed without necessitating an increase in computational power.

Performing analytical transformations of functions on digital systems – especially when determining the characteristics of complex objects of various physical natures – remains an exceptionally challenging and inconvenient task. The successful realization of a combined data-processing method directly depends on the form of the analytical representation of the initial numerical arrays. An analysis of possible approaches to constructing such analytical descriptions reveals that the most suitable method is based on approximating data by segments of orthogonal series employing classical orthogonal polynomials and functions of continuous and discrete arguments [10].

The remarkable approximation properties of orthogonal bases make them highly attractive for solving these problems, while the use of approximation results in various analytical transformations and derivations to obtain quantitative estimates or characteristics renders classical orthogonal bases a promising tool among existing methods and approaches for analytical representation of digital information arrays.

The theory of classical orthogonal bases can be regarded as a generalization of the Fourier series theory to polynomial algebras. Its principal feature lies in the fact that, in most of the defining formulas for specific bases, parameters are present whose variation can substantially alter the properties of orthogonal polynomials and the weight functions that form the given orthogonal basis. This property is particularly significant in problems of optimal analytical approximation, where a given accuracy must be achieved by the shortest possible segment of an orthogonal series. The object of further research will be the development of mathematical models aimed at improving computational techniques and applying these findings to the oil and gas industry.

The solution to the problem of constructing a mathematical framework for fast analytical transformations of functions represented by finite orthogonal series constitutes a cornerstone in the modern development of computer science. As orthogonal bases, systems of functions constructed from classical orthogonal polynomials are considered.

Discrete-variable bases are particularly well-suited for developing software systems – such as diagnostic devices, scanners, and digital analysers – while remaining free from the disadvantages characteristic of piecewise-constant, non-differentiable orthogonal systems (Haar, Walsh, Hadamard). Although such systems are easy to implement on computers, they fail to provide analytical representations of data, thereby excluding analytical methods from the processing pipeline.

References

1. Jassim W.A., Raveendran P., Mukundan R. *New orthogonal polynomials for speech signal and image processing*, IET Signal Processing, **6** (2012), no. 8, 713–723. doi:10.1049/iet-spr.2011.0004

2. Abdulhussain S.H., Ramli A.R., Mahmmod B.M., M. Iqbal Saripan, Al-Haddad S.A.R., Jassim W.A., *A New Hybrid form of Krawtchouk and Tchebichef Polynomials: Design and Application*, J. Math. Imaging and Vision, **6** (2019), 555–570. doi:10.1007/s10851-018-0863-4
3. Bouali A., Elouariachi I., Zahi A., Zenkouar K. *Robust deep image clustering using convolutional autoencoder with separable discrete Krawtchouk and Hahn orthogonal moments*, Intelligent Systems with Applications, **22** (2024), 200387. doi:10.1016/j.iswa.2024.200387
4. Rebiai M., Ould Zmirli M., Bengherbia B., Lachenani S.A. *Faults Diagnosis of Rolling-Element Bearings Based on Fourier Decomposition Method and Teager Energy Operator*, *Arabian Journal for Science and Engineering*, **48** (2023), no 5, 6521–6539. doi:10.1007/s13369-022-07401-4
5. Barmak H.S.A., Flusser J. *New discrete orthogonal moments for signal analysis*, Signal Processing, **141** (2017), 57–73. doi:10.1016/j.sigpro.2017.05.023
6. Thung K.-H., Paramesran R., Lim C.L. *Content-based image quality metric using similarity measure of moment vectors*, Pattern Recognit. **45** (2012), no. 6, 2193–2204.
7. Persi Diaconis, Robert Griffiths, *An introduction to multivariate Krawtchouk polynomials and their applications*, Journal of Statistical Planning and Inference, 154 (2014), 39–53. doi:10.1016/j.jspi.2014.02.004
8. Griffiths B.C. *Orthogonal Polynomials on the Multinomial Distribution*, Australian Journal of Statistics, **13** (1971), no. 1, 27–35. doi:10.1111/j.1467-842X.1971.tb01239.x
9. Rodal J., Area I., Godoy E. *Orthogonal polynomials of two discrete variables on the simplex*, Integral Transforms and Special Functions, 16 (2005), no. 3, 263–280. doi:10.1080/1065246042000272036
10. Rodal J., Area I., Godoy E. *Linear partial difference equations of hypergeometric type: Orthogonal polynomial solutions in two discrete variables*, Journal of Computational and Applied Mathematics, 200 (2007), 722–748. doi:10.1016/j.cam.2006.01.027i
11. Tratnik M.V. *Multivariable Meixner, Krawtchouk, and Meixner–Pollaczek polynomials*, J. Math. Phys. 30 (1989), 27–40. doi:10.1063/1.528507

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ОБРОБКА ЗОБРАЖЕНЬ ЗА ДОПОМОГОЮ ПОЛІНОМІВ КРАВЧУКА: ВІДКРИТЕ ПИТАННЯ

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Проаналізовано можливість побудови спектральних моделей функцій на основі класичних ортогональних поліномів, зокрема поліномів Кравчука, які є дискретними аналогами неперервних ортогональних базисів. Зазначено, що використання таких базисів дозволяє підвищити швидкість та стабільність обчислювальних алгоритмів, зменшити похибку апроксимації та забезпечити аналітичний опис сигналів у цифровій формі. Базиси дискретної змінної є найдоцільнішими при розробленні програмного забезпечення для систем неруйнівного контролю та діагностичних комплексів, оскільки дозволяють уникнути чисельного інтегрування складних функціональних залежностей. Особливу увагу приділено аналітичним властивостям поліномів Кравчука, їх рекурентним співвідношенням та можливості використання у спектральних перетвореннях функцій. Сформульовано задачу про застосування поліномів Кравчука декількох змінних для обробки зображень.

Ключові слова: *поліном Кравчука двох змніних; ортогональний поліном; багаточлен Кравчука.*