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LEMMA ON THE DETERMINANTAL RELATION OF SEQUENCES

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In The On-Line Encyclopedia of Integer Sequences (OEIS), relationships between numerical sequences are frequently cited, although these connections mostly arise spontaneously. Often, the source of such relationships lies in second-order, and less frequently third-order, linear recurrence sequences. This article considers numerical sequences associated with infinite linear recurrence relations and proves a lemma on their determinantal relation. To this end, matrices are introduced that generalize the Hessenberg–Toeplitz matrices. These matrices are associated with unordered partitions of a natural number into natural summands. This general, matrix-based approach to the study of sequences naturally establishes a bijection between mutually conjugate partitions of a natural number into natural summands and provides a framework for identifying connections between sequences associated with linear recurrence relations. The article also presents important corollaries of the lemma and illustrates them by examining the connection between the $n-th$ term of the Pell number sequence and the Fibonacci sequence, as well as their principal determinants.

Key words: *recurrence relations, matrices, determinants, sequences.*

1. Introduction

Numerical sequences arise in various branches of mathematics and the natural sciences, making the further development of methods for their study essential. Often, when analyzing problems, it is possible to construct a linear recurrence relation that connects the sequence under investigation with some known finite or infinite sequence. In such cases, it is worthwhile to study the sequences in pairs, since each complements the other through its properties. Using this approach, determinantal relations between sequences associated by linear recurrence relations can be discovered. This leads to the appearance of new matrices [5] that generalize the Hessenberg–Toeplitz matrices [1], [3], [4]. It is also noteworthy that the determinantal relations between associated sequences are connected with the partitions of a natural number into natural summands [2].

The main result of this paper is Lemma 4.1, which establishes a connection between sequences associated by linear recurrence relations. This lemma provides a powerful matrix-based framework for the study of sequences and polynomials in n variables, which were studied by Euler. These polynomials are known as continuants [7].

2. Partitions and Their Conjugates

Let a positive integer n be expressed as the sum of r positive integers, $n = a_1 + a_2 + \ldots + a_r$. This sum is called a *partition* of the number n, the summands are called the *parts* of the partition, r is the *length* of the partition, and we write the partition in the form

$$a = (a_1, a_2, \ldots, a_r) \vdash n, \tag{2.1}$$

where, by convention, $a_r \leqslant a_{r-1} \leqslant \ldots \leqslant a_2 \leqslant a_1$.

If the partition (2.1) contains λ_1 ones, λ_2 twos, ..., λ_{a_1} parts equal to a_1, then the partition can be written as

$$a = \left(1^{\lambda_1}, 2^{\lambda_2}, \ldots, a_1^{\lambda_{a_1}}\right) \vdash n, \quad \lambda_1 + \lambda_2 + \ldots + \lambda_{a_1} = r.$$

If zero parts are allowed in the partition, then a partition of a non-negative integer n can conveniently be represented as

$$n = \lambda_1 + 2\lambda_2 + \ldots + n\lambda_n.$$

Such a partition is denoted by

$$(1^{\lambda_1}, 2^{\lambda_2}, \ldots, n^{\lambda_n}) \vdash n. \tag{2.2}$$

Here, the sum $\lambda_1 + \lambda_2 + \ldots + \lambda_n$ equals the number of nonzero parts in the partition.

A partition

$$\overline{a} = (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_{a_1}}) \vdash n \tag{2.3}$$

is called the *conjugate* of the partition (2.1) if the following equalities hold:

$$\overline{a_i} = \left|\{a_j : a_j \geqslant i\}\right| = \left|\{j : a_j \geqslant i\}\right|, \quad i = 1, 2, \ldots, a_1; \quad j = 1, 2, \ldots, r.$$

If the equalities $\overline{a_i} = a_i$ hold for all $i = 1, 2, \ldots, r$, then the partition (2.3) is called *self-conjugate*.

For example, the partition $(7,5,5,2,2) \vdash 21$ is conjugate to the partition
$(5,5,3,3,3,1,1) \vdash 21$, and vice versa; while the partition $(4,2,1,1) \vdash 8$ is self-conjugate.

3. Generalized Hessenberg–Toeplitz Matrices

Here, we introduce new matrices that generalize Hessenberg–Toeplitz matrices and significantly simplify the technique of working with sequences.

We consider sequences whose elements belong to a certain field K.

Let us be given two sequences:

$$a : a_{<0} = 0,\, a_0 = -1,\, a_1, a_2, a_3, \ldots$$

and

$$u : u_{<0} = 0,\, u_0 = 1,\, u_1, u_2, u_3, \ldots$$

associated via the linear recurrence relation

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + a_3 u_{n-3} + \ldots \tag{3.1}$$

We construct a square matrix of order n, denoted U_n, whose columns are segments of length n of consecutive terms of the sequence u. Each column of the matrix is arranged such that the indices of its elements increase from top to bottom, and the indices of elements in each row decrease from left to right.

Example 3.1. *An example of a matrix of order four of this type is*

$$U = \begin{pmatrix} u_4 & u_1 & u_0 & u_{-2} \\ u_5 & u_2 & u_1 & u_{-1} \\ u_6 & u_3 & u_2 & u_0 \\ u_7 & u_4 & u_3 & u_1 \end{pmatrix},$$

constructed from segments of length 4 of the sequence

$$u : \ldots, u_{-3}, u_{-2}, u_{-1}, u_0, u_1, u_2, u_3, \ldots$$

Matrices of this type can be uniquely defined either by the elements of their first row, e.g., $U = (u_4, u_1, u_0, u_{-2})_4$, *or by the indices of the elements on the main diagonal:* $U = (4, 2, 2, 1)_4$.

These matrices generalize both Toeplitz and Hessenberg–Toeplitz matrices [3], [4], since, when specified by the indices of their diagonal elements, the matrix $U = (\overbrace{s, \ldots, s}^{n})$ is a Toeplitz matrix, while $U = (\overbrace{1, \ldots, 1}^{n})$ is a Hessenberg–Toeplitz matrix.

Example 3.2. *Here is a matrix of order five:*

$$A_5 = \begin{pmatrix} a_7 & a_4 & a_1 & -1 & 0 \\ a_8 & a_5 & a_2 & a_1 & 0 \\ a_9 & a_6 & a_3 & a_2 & -1 \\ a_{10} & a_7 & a_4 & a_3 & a_1 \\ a_{11} & a_8 & a_5 & a_4 & a_2 \end{pmatrix}_5,$$

constructed from segments of length 5 of the sequence

$$a : \ldots, 0, 0, 0, -1, a_1, a_2, a_3, \ldots$$

This matrix is uniquely defined either by the top elements of its columns, $A = (a_7, a_4, a_1, a_0, a_{-2})_5$, *or by the indices of its diagonal elements,* $A = (7, 5, 3, 3, 2)_5$.

To analyze the sequences, we will use determinants of the matrices. The determinant of matrix U_4 from Example 3.1 is denoted by $d_u(4,3,2,1)$, while the determinant of matrix A_5 from Example 3.2 is denoted analogously by $d_a(7,5,3,3,2)$. In this notation, the symbol d denotes the determinant function, and the subscripts u and a identify the elements from the corresponding matrices.

Such notation is justified because we associate the matrices and their determinants with partitions of a natural number into non-negative integer summands.

The matrix

$$U = (\overbrace{n,\ldots,n}^{n}) = \begin{pmatrix} u_n & u_{n-1} & \cdots & u_1 \\ u_{n+1} & u_n & \cdots & u_2 \\ \vdots & \vdots & \ddots & \vdots \\ u_{2n-1} & u_{2n-2} & \cdots & u_n \end{pmatrix}_n \tag{3.2}$$

is called the *nth principal matrix* of the sequence u, and the determinant

$$d_u(\overbrace{n,\ldots,n}^{n})$$

is its *nth principal determinant*.

For the sequence a, the analogous term *principal determinant* is used.

We also note that columns of matrices formed from sequence elements will be denoted by their top elements in parentheses. For example, the principal matrix (3.2) can be written as $U = ((u_n),(u_{n-1}),\ldots,(u_1))$.

4. Lemma on the Relation Between Numerical Sequences

Lemma 4.1. *Let the sequences*

$$a : a_1,a_2,a_3,\ldots, \quad a_{<0} = 0, \quad a_0 = -1$$

and

$$u : u_1,u_2,u_3,\ldots, \quad u_{<0} = 0, \quad u_0 = 1$$

be related by the recurrence relation (3.1)*, and let*

$$(p_1, p_2, \dots, p_r) \vdash n \quad and \quad (q_1, q_2, \dots, q_s) \vdash n$$

be mutually conjugate partitions. Then the following identity holds:

$$d_u(p_1, p_2, \dots, p_r) = (-1)^{n-s} d_a(q_1, q_2, \dots, q_s), \quad p_1 = s. \tag{4.1}$$

Proof. We prove this lemma for the partition

$$(\overbrace{p, \dots, p}^{r}, \overbrace{q, \dots, q}^{s}) \vdash n = pr + qs, \quad p > q$$

by induction, assuming that for all partitions

$$(\overbrace{p, \dots, p}^{\lambda_1}, \overbrace{q, \dots, q}^{\lambda_2}) \vdash p \cdot \lambda_1 + q \cdot \lambda_2, \quad \lambda_1 \leqslant r, \quad \lambda_2 \leqslant s,$$

the identity (4.1) holds. In the general case, the proof is analogous, but due to the extensive indexing work with matrix elements, the structure of the proof becomes obscured.

We thus consider the determinant

$$d_u(\overbrace{p, \dots, p}^{r}, \overbrace{q, \dots, q}^{s}) =$$

$$= \begin{vmatrix} u_p & u_{p-1} & \cdots & u_{p-r+1} & u_{q-r} & \cdots & u_{q-r-s+1} \\ u_{p+1} & u_p & \cdots & u_{p-r+2} & u_{q-r+1} & \cdots & u_{q-r-s+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ u_{p+r-1} & u_{p+r-2} & \cdots & u_p & u_{q-1} & \cdots & u_{q-s} \\ u_{p+r} & u_{p+r-1} & \cdots & u_{p+1} & u_q & \cdots & u_{q-s+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{p+r+s-1} & u_{p+r+s-2} & \cdots & u_{p+s} & u_{q+s-1} & \cdots & u_q \end{vmatrix}_{r+s} \tag{4.2}$$

1. Using the recurrence relation (3.1), the first column of the determinant (4.2) can be written as:

$$u_p = \left[a_1 u_{p-1} + \ldots + a_{r-1} u_{p-r+1}\right] +$$

$$\left[a_r u_{p-r} + \ldots + a_{p-q+r-1} u_{q-r+1}\right] +$$
$$\left[a_{p-q+r} u_{q-r} + \ldots + a_{p-q+r+s-1} u_{q-r-s+1}\right] +$$
$$\left[a_{p-q+r+s} u_{q-r-s} + \ldots + a_{p+r+s-1} u_{-r-s+1}\right] \quad (4.3)$$

2. Thus, by multiplying the columns of the determinant (4.2) from the second to the rth by $-a_1, \ldots, -a_{r-1}$ respectively, and the columns from $(r+1)$ to $(r+s)$ by $-a_{p-q+r}, \ldots, -a_{p-q+r+s-1}$ respectively, and then adding them to the first column, we obtain a determinant that can be decomposed into a sum of p determinants. We divide these determinants into two groups: A and B. Group A will consist of the $p-q$ determinants whose first columns come from the second bracketed sum in equation (4.3), and group B will consist of the q determinants whose first columns belong to the fourth bracketed sum in that equation. Specifically:

$$A = a_r \left|u_{p-r}, u_{p-1}, \ldots, u_{p-r+1}, u_{q-r}, \ldots, u_{q-r-s+1}\right| + \ldots +$$
$$+ a_{p-q+r-1} \left|u_{q-r+1}, u_{p-1}, \ldots, u_{p-r+1}, u_{q-r}, \ldots, u_{q-r-s+1}\right|;$$
$$B = a_{p-q+r+s} \left|u_{q-r-s}, u_{p-1}, \ldots, u_{p-r+1}, u_{q-r}, \ldots, u_{q-r-s+1}\right| + \ldots +$$
$$+ a_{p+r+s-1} \left|u_{-r-s+1}, u_{p-1}, \ldots, u_{p-r+1}, u_{q-r}, \ldots, u_{q-r-s+1}\right|.$$

3. In each of the determinants that appear in the sums for groups A and B, we rearrange the columns so that the indices of the first row are sorted in decreasing order from left to right.

Since $p-r > q-r$, the first column in each determinant from group A must be moved to the left of column $(r+1)$. This requires $(r-1)$ column swaps. Thus,

$$A = (-1)^{r-1} \left(a_r \left|u_{p-1}, \ldots, u_{p-r+1}, u_{p-r}, u_{q-r}, \ldots, u_{q-r-s+1}\right| + \ldots +\right.$$
$$\left. + a_{p-q+r-1} \left|u_{p-1}, \ldots, u_{p-r+1}, u_{q-r+1}, u_{q-r}, \ldots, u_{q-r-s+1}\right|\right)$$

To reorder all columns in the determinants from group B, the first column in each of them must be moved to the last position. Since each determinant in group B has order $(r+s)$, this requires $(r+s-1)$ column swaps. Therefore,

$$B=(-1)^{r+s-1}\left(a_{p-q+r+s}\left|u_{p-1},\ldots,u_{p-r+1},u_{q-r},\ldots,u_{q-r-s+1},u_{q-r-s}\right|+\ldots\right.$$
$$\left.+a_{p+r+s-1}\left|u_{p-1},\ldots,u_{p-r+1},u_{q-r},\ldots,u_{q-r-s+1},u_{-r-s+1}\right|\right).$$

4. We now express the determinants appearing in both sums in terms of partitions:

$$A=(-1)^{r-1}\left(a_r d_u((p-1)^r,q^s)+\ldots+a_{p-q+r-1}d_u((p-1)^{r-1},q^{s+1})\right);$$

$$B=(-1)^{r+s-1}(a_{p-q+r+s}d_u((p-1)^{r-1},(q-1)^{s+1})+\ldots+$$
$$a_{p+r+s-1}d_u((p-1)^{r-1},(q-1)^s,0)).$$

5. Applying induction, we replace the determinants d_u with d_a by finding the corresponding conjugate partitions:

$$A=(-1)^{r-1}\Big((-1)^{(p-1)r+qs-(p-1)}a_r d_a((r+s)^q,r^{p-q-1})+\ldots+$$
$$+(-1)^{(p-1)(r-1)+q(s+1)-(p-1)}a_{p-q+r-1}d_a((r+s)^q,(r-1)^{p-q-1})\Big)$$
$$=(-1)^{n-p}a_r d_a((r+s)^q,r^{p-q-1})+\ldots$$
$$+(-1)^{n-q+1}a_{r+p-q-1}d_a((r+s)^q,(r-1)^{p-q-1}).$$

$$B=(-1)^{r+s-1}((-1)^{(p-1)(r-2)+(q-1)(s+1)}a_{p-q+r+s}d_a((r+s)^{q-1},$$
$$(r-1)^{p-q})+\ldots+(-1)^{(p-1)(r-1)+(q-1)s-(p-1)}a_{p+r+s-1}d_a((r+s-1)^{q-1},$$
$$(r-1)^{p-q}))=(-1)^{n-q}a_{r+s-(q-p)}d_a((r+s)^{q-1},(r-1)^{p-q})+\ldots+$$
$$+(-1)^{n-1}a_{r+s+p-1}d_a((r+s-1)^{q-1},(r-1)^{p-q}).$$

Therefore,

$$A+B=(-1)^{n-p}\big[a_r d_a((r+s)^q,r^{p-q-1})+\ldots+$$
$$+(-1)^{p-q+1}a_{r+p-q-1}d_a((r+s)^q,(r-1)^{p-q-1})+$$
$$+(-1)^{p-q}a_{r+s-(q-p)}d_a((r+s)^{q-1},(r-1)^{p-q})+\ldots+$$

$$+(-1)^{p-1}a_{r+s+p-1}d_a((r+s-1)^{q-1},(r-1)^{p-q})\Big]. \qquad (4.4)$$

6. Let us write the expression

$$(-1)^{n-p}d_a((r+s)^q,r^{p-q})$$

in matrix form:

$$(-1)^{n-p}\begin{vmatrix} a_{r+s} & \cdots & a_{r+s-(q-1)} & a_{r-q} & \cdots & a_{r-(p-1)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{r+s+q-1} & \cdots & a_{r+s} & a_{r-1} & \cdots & a_{r-(p-q)} \\ a_{r+s+q} & \cdots & a_{r+s+1} & a_r & \cdots & a_{r-(p-q-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{r+s+p-1} & \cdots & a_{r+s-(q-p)} & a_{r+p-q-1} & \cdots & a_r \end{vmatrix} \qquad (4.5)$$

It is not difficult to observe that the expression in square brackets on the right-hand side of equation (4.4) is the expansion of the determinant (4.5) along the elements of the last row. Note that since the matrix of this determinant has order p, the sign of the cofactor of the element $a_{r+s+p-1}$ from the pth row and first column is $(-1)^{p-1}$; the sign of the cofactor of $a_{r+s-(q-p)}$ from the pth row and qth column is $(-1)^{p-q}$; the sign of the cofactor of $a_{r+p-q-1}$ from the pth row and $(q+1)$th column is $(-1)^{p-q-1}$; the sign of the cofactor of a_r is positive.

Thus, the following identity holds:

$$d_u(\overbrace{p,\ldots,p}^{r},\overbrace{q,\ldots,q}^{s}) = (-1)^{n-p}d_a(\overbrace{(r+s),\ldots,(r+s)}^{q},\overbrace{r,\ldots,r}^{p-q}).$$

Finally, we note that in the proof of the lemma, we used only operations within the field K, so the lemma holds for arbitrary elements of this field. □

Consequence 4.1. *If the sequences u and a satisfy the conditions of the lemma, then for their principal determinants the following identity holds:*

$$d_u(\overbrace{n,\ldots,n}^{n}) = d_a(\overbrace{n,\ldots,n}^{n}).$$

The validity of this corollary follows from Lemma 4.1 and the fact that the partition $\overbrace{n,\ldots,n}^{n} \vdash n^2$ is self-conjugate.

Consequence 4.2. *Let $(n \vdash n)$, then the conjugate partition is*

$$(1^n) = (\overbrace{1,\ldots,1}^{n}) \vdash n,$$

and identity (4.1) *takes the form of expressing the nth term of the sequence u via the Hessenberg–Toeplitz matrix of order n:*

$$u_n = \begin{vmatrix} a_1 & -1 & \cdots & 0 & 0 \\ a_2 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_1 & -1 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{vmatrix}. \tag{4.6}$$

Example 4.1. *Let $a_n = F_n$, $n = 1,2,3,\ldots$ be the Fibonacci numbers:*

$$a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5, a_6 = 8, a_7 = 13, a_8 = 21, \ldots,$$

where we assume $a_0 = -1$.

We generate the first few terms of the sequence u using the recurrence relation (3.1)*:*

$$u_1 = 1, \quad u_2 = 2, \quad u_3 = 5, \quad u_4 = 12, \quad u_5 = 29, \quad u_6 = 70, \quad u_7 = 169, \ldots$$

We assume $u_0 = 1$. Thus, we obtain the Pell sequence, whose nth term we denote by P_n.

As a consequence of Corollaries 4.1 and 4.2, we obtain two equalities that hold for any positive integer n. The first equality is:

$$\begin{vmatrix} P_n & P_{n-1} & \cdots & P_1 \\ P_{n+1} & P_n & \cdots & P_2 \\ \vdots & \vdots & \ddots & \vdots \\ P_{2n-1} & P_{2n-2} & \cdots & P_n \end{vmatrix}_n = \begin{vmatrix} F_n & F_{n-1} & \cdots & F_1 \\ F_{n+1} & F_n & \cdots & F_2 \\ \vdots & \vdots & \ddots & \vdots \\ F_{2n-1} & F_{2n-2} & \cdots & F_n \end{vmatrix}_n.$$

This identity demonstrates the deep relationship between the Pell numbers and the Fibonacci numbers.

The second corollary provides a determinantal expression for the nth term of the Pell sequence in terms of the Fibonacci numbers:

$$P_n = \begin{vmatrix} F_1 & -1 & \cdots & 0 & 0 \\ F_2 & F_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{n-1} & F_{n-2} & \cdots & F_1 & -1 \\ F_n & F_{n-1} & \cdots & F_2 & F_1 \end{vmatrix}_n .$$

The author has not found in the scientific literature any determinantal identity linking the Fibonacci and Pell sequences. It is worth noting that, using Lemma 4.1, one can construct arbitrarily many analogous determinantal relations between sequences associated via linear recurrence relations.

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ЛЕМА ПРО ДЕТЕРМІНАНТНИЙ ЗВ'ЯЗОК ПОСЛІДОВНОСТЕЙ

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У The On-Line Encyclopedia of Integer Sequences (OEIS) часто цитуються зв'язки між числовими послідовностями, однак ці зв'язки здебільшого виникають спонтанно. Часто джерелом таких зв'язків є лінійні рекурентні послідовності другого і рідше третього порядків. У даній статті розглядаються асоційовані нескінченним лінійним рекурентним співвідношенням числові послідовності та доведено лему про їх детермінантний зв'язок. З цією метою вводяться матриці, які узагальнюють матриці Гессенберга-Тепліца. Ці матриці ми пов'язуємо із неупорядкованими розбиттями натурального числа на натуральні доданки. Цей загальний, матричний підхід до дослідження послідовностей, природно встановлює бієкцію між взаємно спряженими розбиттями натурального числа на натуральні доданки та дає апарат для встановлення зв'язків між асоційованими лінійним рекурентним співвідношенням послідовностями. У статті також наведено важливі наслідки леми та проілюстровано їх на прикладі зв'язку $n-$того члена послідовності чисел Пелля із послідовністю чисел Фібоначчі.

Ключові слова: *рекурентні співвідношення, матриці, детермінанти, послідовності.*