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ON THE STABILITY OF THE MAXIMUM TERM OF FUNCTIONAL SERIES IN A SYSTEM OF FUNCTIONS

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By L_+ we denote the class of positive continuous on $\mathbb{R}_+ := [0,+\infty)$ functions $l(t)$ such that $l(t) \uparrow +\infty$ $(t \to +\infty)$, and by $\mathscr{W}$ we denote the class of functions $w \in L_+$ such that $\int_1^{+\infty} x^{-2} w(x)dx < +\infty$. The article deals with the functional series of the form $F(x) = \sum_{k=0}^{+\infty} a_k f(x\lambda_k)$, where $\Lambda = (\lambda_k)$ is some sequence of non-negative numbers, $a_k \geq 0$ $(k \geq 0)$, and f is a positive increasing to $+\infty$ function on $[0;+\infty)$ with $f(0) = 1$ and $\ln f(x)$ is a convex function on the same interval. Let us denote $F_w(x) = \sum_{k=0}^{+\infty} a_k e^{w(\lambda_k)} f(x\lambda_k)$,

$$\nu_0(t) = \nu\{u \geq 0\colon\ \ln f(u) \leq t\}, \quad \nu(G) = \sum\nolimits_{\lambda_n \in G} e^{w(\lambda_n)}$$

for every bounded set $G \in \mathbb{R}_+$, where $w \in L_+$. The main result of the paper is the following statement: If there exists a function $w \in L_+$ such that $a_n e^{w(\lambda_n)} f(\lambda_n x) \to 0$ for every $x > 0$, $\ln \nu_0 \in \mathscr{W}$, then there exists a set $E \subset \mathbb{R}_+$ of finite Lebesgue measure such that the asymptotic relation $\ln \mu(x,F) = (1+o(1)) \ln \mu(x,F_w)$ holds as $x \to +\infty$ outside the set E, where $\mu(x,F) = \max\{a_k f(x\lambda_k)\colon k \geq 0\}$.

Key words: *functional series, exceptional set, stability of a maximal term.*

1. Introduction

Let $\mathscr{S}(f,\Lambda)$ be the class of positive convergent for all $x \geq 0$ the functional series of the form

$$F(x) = \sum_{k=0}^{+\infty} a_k f(x\lambda_k) \tag{1}$$

and $\mathscr{S}_+(f_0,\Lambda)$ be the class of positive convergent for all $x \geq 0$ the functional series of the form

$$F(x) = \sum_{k=0}^{+\infty} a_k f_0(x+\lambda_k) \tag{2}$$

such that $a_k \geq 0$ $(k \in \mathbb{Z}_+)$; here $\Lambda = (\lambda_k)$ is some sequence of the nonnegative numbers $\lambda_k \geq 0$ $(k \geq 0)$, such that $\lambda_k \neq \lambda_j$ for all $k \neq j$; f and f_0 are some positive functions such that the functions $\ln f(x)$ and $\ln f_0(x)$ are convex functions on $[0,+\infty)$. In the case $f(x) \equiv e^x$, we obtain a Dirichlet series of the form

$$F_1(x) = \sum_{k=0}^{+\infty} a_k e^{x\lambda_k}, \tag{3}$$

that converges for all $x \geq 0$, and we will write $F_1 \in \mathscr{D}(\Lambda) = \mathscr{S}(f,\Lambda)$ with $f(x) = e^x$.

Let L be a class of positive continuous on $\mathbb{R}_+ := [0,+\infty)$ the functions $l(t)$ such that $l(t) \to +\infty$ $(t \to +\infty)$. By L_+ we denote the subclass of L such that $l(t) \uparrow +\infty$ as $x \to +\infty$, and by $\mathscr{W}$ the class of functions $w \in L_+$ such that

$$\int_1^{+\infty} x^{-2} w(x)dx < +\infty.$$

For a series $F \in \mathscr{S}(f,\Lambda)$ and any sequence (b_n), $b_n \in \mathbb{R}_+ \setminus \{0\}$ $(n \geq 0)$ we consider

$$B^+(x) = \sum_{n=0}^{+\infty} a_n b_n f(x\lambda_n), \quad B^-(x) = \sum_{n=0}^{+\infty} a_n b_n^{-1} f(x\lambda_n).$$

Proposition 1.1. *If a sequence* $\{b_n :\ n \geqslant 0\} \subset \mathbb{R}_+ \setminus \{0\}$ *satisfies condition*

$$b = \varlimsup_{n\to+\infty} \frac{\max\{\ln b_n, -\ln b_n\}}{\ln f(\lambda_n)} < +\infty, \tag{4}$$

then $F \in \mathscr{S}(f,\Lambda) \Longleftrightarrow B^+ \in \mathscr{S}(f,\Lambda) \Longleftrightarrow B^- \in \mathscr{S}(f,\Lambda)$.

Proof. From condition (4)

$$\max\{\ln b_n, -\ln b_n\} \le b_* \ln f(\lambda_n) \quad (n \ge 1) \tag{5}$$

for some $b_* < +\infty$. If $F \in \mathscr{S}(f,\Lambda)$, then by the convexity of the function $\ln f(x)$ we get

$$\frac{\ln f(2x\lambda_n) - \ln f(x\lambda_n)}{x\lambda_n} \ge \frac{\ln f(x\lambda_n)}{x\lambda_n} \ge \frac{\ln f(\lambda_n)}{\lambda_n} \tag{6}$$

for all $x \ge 1$. Hence, for arbitrary fixed $b_* < +\infty$ one has

$$b_* \ln f(\lambda_n) + \ln f(x\lambda_n) \le \ln f(2x\lambda_n) \quad (x \ge b_*). \tag{7}$$

Therefore, combining (5) and (7) we deduce

$$b_n a_n f(x\lambda_n) \le a_n f(2x\lambda_n) \quad (x \ge b_*).$$

So, $F \in \mathscr{S}(f,\Lambda) \Longrightarrow B^+ \in \mathscr{S}(f,\Lambda)$.

Let us prove the reverse implication $B^+ \in \mathscr{S}(f,\Lambda) \Longrightarrow F \in \mathscr{S}(f,\Lambda)$. Using inequalities (5) and (7), we obtain

$$a_n f(x\lambda_n) \le b_n a_n \exp\{\ln f(x\lambda_n) + b_* \ln f(\lambda_n)\} \le b_n a_n f(2x\lambda_n) \quad (x \ge b_*),$$

hence, again by condition (5), we have $B^+ \in \mathscr{S}(f,\Lambda) \Longrightarrow F \in \mathscr{S}(f,\Lambda)$.

From the proved implications, we easily obtain that

$$B^+ \in \mathscr{S}(f,\Lambda) \Longleftrightarrow B^- \in \mathscr{S}(f,\Lambda).$$

□

By $\mathscr{S}_*(f,\Lambda)$ we denote the class of formal series of form (1) such that $a_n f(x\lambda_n) \to 0$ $(n \to +\infty)$ for every $x \in \mathbb{R}_+$, i.e., for every $x \in \mathbb{R}_+$ there exists the maximal term

$$\mu(x,F) = \max\{|a_n| f(x\lambda_n)\colon n \geqslant 0\} < +\infty.$$

We also write $\mathscr{D}_*(\Lambda) = \mathscr{S}_*(f,\Lambda)$ with $f(x) = e^x$. Clearly, $\mathscr{S}(f,\Lambda) \subset \mathscr{S}_*(f,\Lambda)$.

Proposition 1.2. *If* $F \in \mathscr{S}_*(f,\Lambda)$ *and the condition*

$$h := \varliminf_{n\to+\infty} \frac{-\ln a_n}{\ln n} > 1 \tag{8}$$

or the condition

$$\tau_f(\Lambda) := \varlimsup_{n\to+\infty} \frac{\ln n}{\ln f(\lambda_n)} < +\infty \tag{9}$$

holds, then $F \in \mathscr{S}(f,\Lambda)$.

Proof. It is sufficient to prove the convergence of a series (1) for all enough large x. At first, we assume that condition (9) is satisfied. From the conditions $F \in \mathscr{S}_*(f,\Lambda)$ and (9) it follows that for given x the inequalities $a_n f(2x\lambda_n) \le 1$ and $\ln n < \tau \ln f(\lambda_n)$ with some $\tau \in (\tau_f(\Lambda, +\infty))$ hold for all $n \ge n_0$. Therefore, for $x \ge 2\tau$ inequality (6) implies

$$a_n f(x\lambda_n) = a_n f(2x\lambda_n) \exp\{-(\ln f(2x\lambda_n) - \ln f(x\lambda_n))\} \le$$
$$\le \exp\{-\ln f(x\lambda_n)\} \le \exp\{-x \ln f(\lambda_n)\} \le \exp\{-2\ln n\}.$$

So, series (1) is convergent for all $x \ge 2\tau$.

Assume that condition (8) is satisfied. Similarly to inequality (6) we have

$$\frac{\ln f(Kx\lambda_n) - \ln f(x\lambda_n)}{(K-1)x\lambda_n} \ge \frac{\ln f(x\lambda_n)}{x\lambda_n}, \quad K > 1,$$

so $\ln f(Kx\lambda_n) \ge K \ln f(x\lambda_n)$. From the conditions $F \in \mathscr{S}_*(f,\Lambda)$ and inequality (6) it follows again that for given x

$$\ln a_n + x \ln f(\lambda_n) \to -\infty \quad (n \to +\infty).$$

Since, by condition (8), $\ln a_n \le -h_* \cdot \ln n$ for arbitrary $h_* \in (1,h)$ and for enough large $n \ge n_0$, we obtain

$$\ln a_n + \ln f(x\lambda_n) \le (1 - \frac{1}{K}) \ln a_n \le -(1 - \frac{1}{K}) \cdot h_* \cdot \ln n.$$

Let us now choose $K > 1$ so that $h_1 := (1 - \frac{1}{K}) \cdot h_* > 1$. Then

$$\sum_{n=n_0}^{+\infty} a_n f(x\lambda_n) \le \sum_{n=n_0}^{+\infty} n^{-h_1} < +\infty,$$

that is $F \in \mathscr{S}(f,\Lambda)$. □

2. Main result.

We call that a series of the form (1) (maximal tem of the series) **is stable** if the relations

$$\ln\mu(x,F) = (1+o(1))\ln\mu(x,B^+) = (1+o(1))\ln\mu(x,B^-) \qquad (10)$$

hold as $x\to+\infty$ outside some set $E\subset[0,+\infty)$ of the finite Lebesgue measure, i.e. $\operatorname{meas}E := \int_E dx < +\infty$.

For a function $w\in L$ let us denote

$$B_w(x) = \sum_{n=0}^{+\infty} a_n e^{w(\lambda_n)} f(x\lambda_n).$$

From Theorem 2 and Theorem 3 ([4]), proved for entire multiple Dirichlet series, it follows the following statement.

Theorem A ([4], Theorem 2). *Let $w\in L$, $B_w\in D_*(\lambda)$ and condition*

$$\int_0^{+\infty} t^{-2}\ln\nu_0(t)dt < +\infty \qquad (11)$$

is satisfied, where $\nu_0(t) = \int\limits_0^t e^{w(x)}dn(x)$, $n(x) = \sum_{\lambda_n\leqslant x} 1$. Then relation

$$\ln\mu(x,F) = (1+o(1))\ln\mu(x,B_w) \qquad (12)$$

holds as $x\to+\infty$ outside some set $E\subset[0;+\infty)$, $\operatorname{meas}E < +\infty$.

Theorem A implies the following corollary.

Corollary 2.1. *Let $F\in\mathscr{D}_*(\Lambda)$. If there exists a function $w\in L$ such that $F_w\in\mathscr{D}_*(\Lambda)$, $\ln\nu\in\mathscr{W}$ (here $\nu(t) = \sum_{\lambda_n\leq t} e^{w(\lambda_n)}$) and*

$$e^{-w(\lambda_n)} \leqslant b_n \leqslant e^{w(\lambda_n)} \quad (n\geq k_1), \qquad (13)$$

then there exists a set $E\subset\mathbb{R}_+$ of finite Lebesque measure such that

$$\ln\mu(x,F) = (1+o(1))\ln\mu(x,B_+) = (1+o(1))\ln\mu(x,B_-) \qquad (14)$$

as $x\to+\infty$ $(x\notin E)$.

Other versions of these statements were proved earlier in paper [2] by O.B. Skaskiv and O.M. Trakalo.

Let us denote

$$\nu_0(t) = \nu\{u \geq 0\colon\ \ln f(u) \leq t\}, \quad \nu(G) = \sum\nolimits_{\lambda_n \in G} e^{w(\lambda_n)}$$

for every bounded set $G \in \mathbb{R}_+$. In this note, we will first prove the following theorem.

Theorem 2.1. *Let $F \in \mathscr{S}_*(f,\Lambda)$. If there exists a function $w \in L_+$ such that $B_w \in \mathscr{S}_*(f,\Lambda)$, $\ln \nu_0 \in \mathscr{W}$ and inequalities* (13) *are valid, then there exists a set $E \subset \mathbb{R}_+$ of finite Lebesque measure such that relation* (14) *holds as $x \to +\infty$ $(x \notin E)$.*

We need the following statement from [3, Corollary 1]. We consider the class $\mathscr{I}(\nu, f)$ of functions $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ represented by integrals of the form

$$F(x) = \int_0^{+\infty} g(t) f(tx) \nu(dt),$$

where ν is a locally finite measure on $\mathbb{R}_+$, g is positive ν-measurable function, f is positive increasing to $+\infty$ in $[0;+\infty)$ function such that $f(0) = 1$ and $\ln f(x)$ is a convex on the interval $[0;+\infty)$ function.

Lemma 2.1 ([3]). *If condition* (11) *holds with $\nu_0(t) = \nu\big(\{u \geq 0\colon\ \ln f(u) \leq t\}\big)$, then for every function $F \in \mathscr{I}(\nu, f)$ there exists a set E of finite Lebesgue measure such that the asymptotic relation*

$$\ln F(x) \leqslant (1 + o(1)) \ln \mu(x, F) \tag{15}$$

holds as $x \to +\infty$ $(x \notin E)$, where $\mu(x,F) = \sup\{g(t)f(tx)\colon x \in \operatorname{supp}\nu\}$ and $\operatorname{supp}\nu$ is the support of the measure ν.

Proof of Theorem 2.1. Note that relation (13) will follow from the fact that

$$\ln \mu(x,F) = (1 + o(1)) \ln \mu(x, B_w) \tag{16}$$

as $x \to +\infty$ outside of some set E of finite Lebesgue measure. Let us prove relation (16).

Let $a(t), b(t)$ be measurable nonnegative functions on $\mathbb{R}_+$ such that $a(\lambda_n) = a_n$, $b(\lambda_n) = e^{w(\lambda_n)}$ and

$$\mu(x,F) = \sup\{a(t)f(tx) \colon t \in \mathbb{R}_+\}, \ \ \mu(x,B_w) = \sup\{a(t)b(t)f(tx) \colon t \in \mathbb{R}_+\}.$$

It is enough to take that $a(t) = 0$ for $t \notin \{\lambda_n \colon n \in \mathbb{Z}_+\}$.

Then for all $x \in \mathbb{R}$ we get

$$\mu(x,F) \leqslant \mu(x,B_w) \leqslant B_w(x) = \sum_{n=0}^{+\infty} a_n b(\lambda_n) f(x\lambda_n) = \int\limits_{\mathbb{R}_+} a(t) f(tx) \nu(dt), \quad (17)$$

where measure ν is such that $\nu(G) = \sum_{n=0}^{+\infty} b(\lambda_n)\delta_{\lambda_n}(G)$ for each bounded set $G \subset \mathbb{R}_+$ and $\delta_\lambda(G) = 1$ for $\lambda \in G$ and $\delta_\lambda(G) = 0$ for $\lambda \notin G$.

From condition $\ln \nu_0 \in \mathscr{W}$ we immediately get that condition (11) of Lemma 2.1 is satisfied. Applying Lemma 2.1 to the integral in (17), as $x \to +\infty$ $(x \notin E)$, (here a set E is such as in Lemma 2.1) we obtain

$$\ln \mu(x,F) \leqslant \ln \mu(x,B_w) \leqslant (1+o(1)) \ln \mu_*(x),$$

where $\mu_*(x) = \max\{a(t)f(xt) \colon t \in \mathbb{R}_+\}$. As for the choice of function $a(t)$ we get $\mu_*(x) = \mu(x,F)$ and deduce relation (16). □

3. Conjectures.

1. A statement similar to Lemma 2.1 is true for integrals of the form

$$F(x) = \int\limits_0^{+\infty} g(t) f(x+t) \nu(dt)$$

(see, for example, the proof of Theorem 2 in [5].)

2. A statement similar to Theorem 2.1 is true for series of form (2).

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ПРО СТІЙКІСТЬ МАКСИМАЛЬНОГО ЧЛЕНА ФУНКЦІОНАЛЬНОГО РЯДУ ЗА СИСТЕМОЮ ФУНКЦІЙ

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Через L_+ *позначимо клас додатних неперервних на* $\mathbb{R}_+ := [0,+\infty)$ *функцій* $l(t)$ *таких, що* $l(t)\uparrow+\infty$ $(t\to+\infty)$, *а через* $\mathscr{W}$ *– клас функцій* $w\in L_+$ *таких, що* $\int_1^{+\infty} x^{-2}w(x)dx<+\infty$. *Розглядаються функціональні ряди вигляду* $F(x)=\sum\limits_{k=0}^{+\infty} a_k f(x\lambda_k)$, *де* $\Lambda=(\lambda_k)$ *— послідовність невід'ємних чисел,* $a_k\geq 0$ $(k\geq 0)$, f *— додатна функція, що зростає до* $+\infty$ *на* $[0;+\infty)$ *і* $f(0)=1$, *а функція* $\ln f(x)$ *— опукла на інтервалі* $[0;+\infty)$. *Позначимо* $F_w(x)=\sum_{k=0}^{+\infty} a_k e^{w(\lambda_k)} f(x\lambda_k)$,

$$\nu_0(t)=\nu\{u\geq 0\colon\ \ln f(u)\leq t\},\quad \nu(G)=\sum\nolimits_{\lambda_n\in G} e^{w(\lambda_n)}$$

для кожної обмеженої множини $G \in \mathbb{R}_+$, *де* $w \in L_+$. *Основним результатом статті є таке твердження: якщо існує така функція* $w \in L_+$, *що* $a_n e^{w(\lambda_n)} f(\lambda_n x) \to 0$ *для кожного* $x > 0$, $\ln \nu_0 \in \mathscr{W}$, *то існує множина* $E \subset \mathbb{R}_+$ *скінченної міри Лебега така, що асимптотичне співвідношення* $\ln \mu(x,F) = (1+o(1)) \ln \mu(x,F_w)$ *виконується при* $x \to +\infty$ *зовні множини* E, *де* $\mu(x,F) = \max\{a_k f(x\lambda_k) \colon k \geq 0\}$.

Ключові слова: *функціональні ряди, виняткова множина, стійкість максимального члена.*