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PAR-FUNCTIONS OF TOEPLITZ MATRICES AND PARTITION POLYNOMIALS

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This paper explores certain properties of the par and par$^+$ functions of Toeplitz matrices. These functions are studied in tandem due to their many shared properties. Since the combinatorial foundation of these functions lies in ordered partitions of a natural number into non-negative integers, it becomes possible to represent them as partition polynomials and to construct recursive algorithms for their computation. In addition to a brief introduction to these functions, the paper presents a recurrence relation for computing the par-functions of Toeplitz matrices, which enables the unification of a broad class of linear recurrence relations. As linear recurrence relations are often related to partition polynomials, the representation of these functions as partition polynomials is also studied. The article includes an example that utilizes the fact that the multilinear polynomials of par$^+$ and par-functions of square matrices contain 2^{n-1} terms, with the par-function comprising half positive and half negative terms. Two combinatorial identities are derived using a Toeplitz matrix whose entries are all equal to one.

Key words: *par-functions, Toeplitz matrices, partition polynomials.*

1. Introduction

Recurrence relations permeate various branches of mathematics and provide efficient polynomial algorithms for computing mathematical objects.

For the determinant and permanent functions of square matrices, general recurrence relations have not yet been constructed. Moreover, no polynomial-time algorithm has been developed for computing the permanent of a square matrix. A general representation of these functions in terms of partition polynomials is also unknown. In [1], a function of square matrices was introduced whose structure allows it to be computed using linear recurrence relations. This makes it possible to describe a broad class of linear recurrence relations using par-functions.

We provide a brief introduction to the par-function of a square matrix [1]. This function is constructed to satisfy two classical conditions, which are also fulfilled by the determinant:

1) Every entry of the matrix influences the value of the par-function;

2) Half of the terms in the multilinear polynomial of the par-function have a positive sign, and the other half have a negative sign.

Let

$$A_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{1.1}$$

be a square matrix with entries from an arbitrary field K.

Let $(n_1, n_2, \ldots, n_r) \vdash n$ be an r-partition belonging to the set of all ordered partitions $C(n,+)$ of a natural number n into non-negative integers. We define a bijection between the elements of $C(n,+)$ and a subset $CS_n \subset S_n$, where S_n is the set of all permutations of the first n natural numbers, via the following algorithm:

$$\begin{gathered}(n_1, n_2, \ldots, n_r) \vdash n \longleftrightarrow \\ (n_1, n_1 - 1, \ldots, 1; n_1 + n_2, n_1 + n_2 - 1, \ldots, n_1 + 1; \ldots; n_1 + \ldots + n_r, \\ n_1 + \ldots + n_r - 1, \ldots, n_1 + \ldots + n_{r-1} + 1)_r.\end{gathered} \tag{1.2}$$

We note that the set CS_n is defined by the right-hand side of the bijection (1.2).

The par-function and the par^+-function of matrix (1.1) were introduced in [1], where an analogy with determinants and permanents of square matrices is maintained. These functions are defined by the following formulas:

$$par(A_n) = \sum_{r=1}^{n} \sum_{(i_1,i_2,\ldots,i_n)_r \in CS_n} (-1)^{n-r} a_{1i_1} a_{2i_2} \cdots a_{ni_n}; \qquad (1.3)$$

$$\mathrm{par}^+(A_n) = \sum_{r=1}^{n} \sum_{(i_1,i_2,\ldots,i_n)_r \in CS_n} a_{1i_1} a_{2i_2} \cdots a_{ni_n}. \qquad (1.4)$$

It is shown in [1, Theorem 1.1] that the par-function (1.3) satisfies the recurrence relation:

$$par(A_n) = \sum_{i=1}^{n} \prod_{j=0}^{i-1} (-1)^{i-1} a_{n-j,n-i+j+1} par(A_{n-i}), \qquad (1.5)$$

where $par(A_0) = 1$ and $par(A_{<0}) = 0$.

A recurrence relation for the par^+-function (1.4) can be derived using the following theorem:

Theorem 1.1. *[1] Let*

$$A_n^{'} = \begin{pmatrix} a_{11} & -a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & -a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & -a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}$$

and

$$A_n = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}.$$

Then the following identity holds:

$$\mathrm{par}(A_n^{'}) = \mathrm{par}^+(A_n).$$

This gives the recurrence:

$$par^{+}(A_n) = \sum_{i=1}^{n} \prod_{j=0}^{i-1} a_{n-j,n-i+j+1} par^{+}(A_{n-i}), \tag{1.6}$$

where $par^{+}(A_0) = 1$ and $par^{+}(A_{<0}) = 0$.

2. Recurrence Relations for the par-Functions of Toeplitz Matrices

First, we note that for functions of Toeplitz matrices that are not Hessenberg matrices, no general recurrence relations have yet been constructed for their evaluation. The author is also unaware of any representations of these functions as partition polynomials.

We consider a Toeplitz matrix [2]:

$$T_n = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{1-n} \\ t_1 & t_0 & \cdots & t_{2-n} \\ \vdots & \vdots & \cdots & \vdots \\ t_{n-1} & t_{n-2} & \cdots & t_0 \end{pmatrix} \tag{2.1}$$

with entries from some field.

Proposition 2.1. *For the matrix (2.1), the following recurrence relations hold:*

$$\begin{gathered} \mathrm{par}(T_n) = \sum_{i=1}^{n} \prod_{j=0}^{i-1} (-1)^{i-1} t_{i-2j-1} \mathrm{par}(T_{n-i}) = \\ t_0 \mathrm{par}(T_{n-1}) - t_1 t_{-1} \mathrm{par}(T_{n-2}) + t_2 t_0 t_{-2} \mathrm{par}(T_{n-3}) - \\ t_3 t_1 t_{-1} t_{-3} \mathrm{par}(T_{n-4}) + t_4 t_2 t_0 t_2 t_4 \mathrm{par}(T_{n-5}) - \\ \mathrm{t}_5 t_3 t_1 t_{-1} t_{-3} t_{-5} \mathrm{par}(T_{n-6}) + t_6 t_4 t_2 t_0 t_{-2} t_{-4} t_{-6} \mathrm{par}(T_{n-7}) - \ldots, \end{gathered} \tag{2.2}$$

where $\mathrm{par}(T_0) = 1$ *and* $\mathrm{par}(T_{<0}) = 0$*;*

$$\mathrm{par}^{+}(T_n) = \sum_{i=1}^{n} \prod_{j=0}^{i-1} t_{i-2j-1} \mathrm{par}^{+}(T_{n-i}) =$$

$$t_0\mathrm{par}^+(T_{n-1})+t_1t_{-1}\mathrm{par}^+(T_{n-2})+t_2t_0t_{-2}\mathrm{par}^+(T_{n-3})+$$
$$t_3t_1t_{-1}t_{-3}\mathrm{par}^+(T_{n-4})+t_4t_2t_0t_2t_4\mathrm{par}^+(T_{n-5})+ \tag{2.3}$$
$$\mathrm{t}_5t_3t_1t_{-1}t_{-3}t_{-5}\mathrm{par}^+(T_{n-6})+t_6t_4t_2t_0t_{-2}t_{-4}t_{-6}\mathrm{par}^+(T_{n-7})-\ldots,$$

where $\mathrm{par}^+(T_0)=1$ *and* $\mathrm{par}^+(T_{<0})=0$.

Proof. Let us assume in matrix (1.1) that $a_{ij}=t_{i-j}$ for $i,j=1,2,\ldots,n$. Then the matrix has the structure of a Toeplitz matrix as in (2.1). Substituting $a_{ij}=t_{i-j}$ into the recurrence relation (1.5), we obtain the relation:

$$par(T_n)=\sum_{i=1}^{n}\prod_{j=0}^{i-1}(-1)^{i-1}t_{i-2j-1}par(T_{n-i}).$$

The product $\prod_{j=0}^{i-1}(-1)^{i-1}t_{i-2j-1}$ for even i becomes:

$$-(t_1t_{-1})(t_3t_{-3})\cdots(t_{i-1}t_{-(i-1)}),$$

while for odd i it consists of an odd number of terms:

$$t_0(t_2t_{-2})(t_4t_{-4})\cdots(t_{i-1}t_{-(i-1)}),$$

which proves the identity (2.2). Identity (2.3) is proved similarly, omitting the signs $(-1)^{i-1}$. □

3. Partition Polynomials of par-Functions for the Toeplitz Matrix

In [1], a connection was established between the determinant of a Hessenberg matrix [3] and the par-function of a square matrix. However, for the determinants and permanents of Hessenberg matrices, there are known representations in terms of partition polynomials.

If

$$H=\begin{pmatrix} h_1 & 1 & 0 & \cdots & 0\\ h_2 & h_1 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ h_{n-1} & h_{n-2} & h_{n-3} & \cdots & 1\\ h_n & h_{n-1} & h_{n-2} & \cdots & h_1 \end{pmatrix},$$

then the determinant and permanent of this matrix have the following partition polynomial representations:

$$\det(H)=\sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n}(-1)^{n-(\lambda_1+\lambda_2+\ldots+\lambda_n)}\frac{(\lambda_1+\lambda_2+\ldots+\lambda_n)}{\lambda_1!\lambda_2!\cdots\lambda_n!}h_1^{\lambda_1}h_2^{\lambda_2}\cdots h_n^{\lambda_n};$$

$$\operatorname{per}(H)=\sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n}\frac{(\lambda_1+\lambda_2+\ldots+\lambda_n)}{\lambda_1!\lambda_2!\cdots\lambda_n!}h_1^{\lambda_1}h_2^{\lambda_2}\cdots h_n^{\lambda_n}.$$

For the par-functions of Toeplitz matrices, we have the equalities

$$h_i=t_{i-1}t_{i-3}\cdots t_{3-i}t_{1-i},\quad i=1,2,\ldots,n,$$

$$h_i=c_{i-1}c_{i-3}\cdots c_{n-(i-3)}c_{n-(i-1)},\quad i=1,2,\ldots,n,$$

therefore, the following statement holds

Proposition 3.1. *For the par and par$^+$-functions of the Toeplitz matrix (2.1), the following partition polynomial representations are valid:*

$$par(T_n)=\sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n}(-1)^{n-(\lambda_1+\lambda_2+\ldots+\lambda_n)}\frac{(\lambda_1+\lambda_2+\ldots+\lambda_n)}{\lambda_1!\lambda_2!\cdots\lambda_n!}\times$$
$$\times\prod_{i=1}^{n}(t_{i-1}t_{i-3}\cdots t_{3-i}t_{1-i})^{\lambda_i},$$

Analogously, for the par$^+$-function, we have:

$$par^+(T_n)=\sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n}\frac{(\lambda_1+\lambda_2+\ldots+\lambda_n)}{\lambda_1!\lambda_2!\cdots\lambda_n!}\prod_{i=1}^{n}(t_{i-1}t_{i-3}\cdots t_{3-i}t_{1-i})^{\lambda_i}.$$

The last two expressions can be rewritten in a more convenient form for practical applications, taking into account that

$$\prod_{i=1}^{n}(t_{i-1}t_{i-3}\cdots t_{3-i}t_{1-i})=$$

$$\begin{cases} t_0^{\lambda_1+\lambda_3+\ldots+\lambda_{n-1}}(t_1t_{-1})^{\lambda_2+\lambda_4+\ldots+\lambda_n}(t_2t_{-2})^{\lambda_3+\lambda_5+\ldots+\lambda_{n-1}}\cdots, & \text{if n is even,}\\ t_0^{\lambda_1+\lambda_3+\ldots+\lambda_n}(t_1t_{-1})^{\lambda_2+\lambda_4+\ldots+\lambda_{n-1}}(t_2t_{-2})^{\lambda_3+\lambda_5+\ldots+\lambda_n}\cdots, & \text{if n is odd.}\end{cases}$$

Example 3.1. *Let all entries of the matrix (2.1) be equal to one. Then the value of the par-function of this Toeplitz matrix can be computed using the recurrence:*

$$par(T_n) = par(T_{n-1}) - par(T_{n-2}) + par(T_{n-3}) - \ldots + (-1)^{n-1} par(T_0),$$

with initial conditions $par(T_{<0}) = 0$ *and* $par(T_0) = 1$. *By induction, we have* $par(T_i) = 0$ *for* $i > 1$.

Thus, the corresponding partition polynomial identity for the par-function of this specific matrix is:

$$\sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^{n-(\lambda_1+\lambda_2+\ldots+\lambda_n)} \frac{(\lambda_1+\lambda_2+\ldots+\lambda_n)}{\lambda_1!\lambda_2!\cdots\lambda_n!} 1^{\lambda_1} 1^{\lambda_2} \cdots 1^{\lambda_n} = 0.$$

For the par^+*-function of this unit Toeplitz matrix, the recurrence relation is:* $par^+(T_n) = par^+(T_{n-1}) + par^+(T_{n-2}) + par^+(T_{n-3}) + \ldots + par^+(T_0)$, *where* $par^+(T_{<0}) = 0$ *and* $par^+(T_0) = 1$. *It is clear that* $par^+(T_n) = 2^{n-1}$. *Therefore, we obtain the identity:*

$$\sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} \frac{(\lambda_1+\lambda_2+\ldots+\lambda_n)}{\lambda_1!\lambda_2!\cdots\lambda_n!} 1^{\lambda_1} 1^{\lambda_2} \cdots 1^{\lambda_n} = 2^{n-1}.$$

The first identity in this example confirms that the multilinear polynomial of the par-function of a square matrix (1.1) contains equal numbers of positive and negative terms. The second confirms that the multilinear polynomials of both functions consist of 2^{n-1} *terms.*

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PAR-ФУНКЦІЇ МАТРИЦЬ ТЕПЛІЦА ТА МНОГОЧЛЕНИ РОЗБИТТІВ

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У статті досліджуються деякі властивості par та par^{+}-функцій матриць Тепліца. Ці функції досліджуються у парі, бо мають багато спільних властивостей. Оскільки комбінаторною основою цих функцій є впорядковані розбиття натурального числа на цілі невід'ємні доданки, то стало можливим їх подання у вигляді многочленів розбиттів та будувати рекурсивні алгоритми їх обчислень. У статті, крім короткого ознайомлення з цими функціями, будується рекурентне співвідношення для обчислення par-функцій матриць Тепліца, яке дозволяє об'єднати великий клас лінійних рекурентних співвідношень. Оскільки лінійні рекурентні співвідношення часто пов'язані із многочленами розбиттів, то у статті також досліджується подання цих функцій у вигляді многочленів розбиттів. У статті також наведено приклад, у якому використовується той факт, що полілінійні поліноми par^{+} та par-функцій квадратних матриць містять 2^{n-1} доданків, а par-функція містить половину додатних і половину від'ємних доданків. Для виведення двох комбінаторних рівностей використано матрицю Тепліца, всі елементи якої є одиницями.

Ключові слова: *par-функції, матриці Тепліца, многочлени розбиттів.*