МАТЕМАТИКА ТА МЕХАНІКА

Математичний аналіз

УДК 517.537.72+517.548.9
DOI: 10.31471/2304-7399-2024-19(73)-9-17

ON THE FUNCTIONS RELATED WITH THE ANALYTIC SOLUTIONS OF THE CAUCHY PROBLEM FOR WAVE AND HEAT EQUATIONS

A.I. Bandura
Ivano-Frankivksk National Technical University of Oil and Gas;
76019, 15 Karpatska street, Ivano-Frankivsk, Ukraine;
e-mail: andriykopanytsia@gmail.com

We investigate properties of entire solutions of the Cauchy problem for one-dimensional homogeneous hyperbolic equation. Considering analytic continuation of the solutions given by the D'Alambert formula we have found some conditions providing L-index boundedness in the direction for some functions related with the solutions. In particular, for homogomogeneous wave equation $c^2\frac{\partial^2}{\partial x^2}u(x,t)=\frac{\partial^2}{\partial t^2}u(x,t)$ *with initial conditions* $u(x,0)=\varphi(x)$, $u_t(x,0)=\psi(x)$ *its solution has the form*

$$u(x,t)=\frac{\varphi(x+ct)+\varphi(x-ct)}{2}+\frac{1}{2c}\int\limits_{x-ct}^{x+ct}\psi(\alpha)d\alpha.$$

We study the functions $\mathfrak{H}(x,t)=\frac{\varphi(x+ct)+\varphi(x-ct)}{2}+\frac{\mathfrak{C}}{2c}\int\limits_{x-ct}^{x+ct}\psi(\alpha)d\alpha$, *where* x, t, $c\in\mathbb{C}$, $\mathfrak{C}$ *is a positive constant which is determined with some conditions by the functions* φ *and* ψ*. Our main result gives sufficient conditions of boundedness of* $\mathfrak{L}$*-index in a direction* $\mathbf{b}$ *for the functions* $\mathfrak{H}$*. Its proof uses known sufficent conditions for the sum of entire functions. At the* *end, we pose open problems concerning conditions of the directional L-index boundedness for analytic solutions of the Cauchy problem of the heat equation. The conditions will allow a qualitative description of local and asymptotic behavior of the parabolic equation analytic solutions presenting the temperature distribution in the process of plasma electrolytic oxidation.*

Key words: *analytic solution, Cauchy problem, one-dimensional hyperbolic equation, sum of functions, bounded L-index in a direction, wave equation, heat equation, PEO.*

1. Introduction

In the last years, there were published many papers on analytic solutions of partial and directional differential equations, and their properties such as local behavior of maximum modulus on the disc, polydics, ball, some uniform distribution of zeros, partial and directional logarithmic derivatives outside some exceptional sets, grotwth estimates, etc. These properties are justified by methods of theory of functions having bounded index. These investigations deal general solutions of the equations. But for many partial differential equations it is very difficult to study the asymptotic and local behavior of solutions (see below example from Goldberg [2]) bacuse they have a sufficiently general structure.

It is known that every entire or meromorphic solution of an ordinary algebraic differential equation in the complex plane has finite order of growth. This does not hold for algebraic partial differential equations. A. Golberg presented such an example [2]

$$z_1 \frac{\partial w}{\partial z_1} - z_2 \frac{\partial w}{\partial x_2} = 0.$$

It is satisfied by the function $w = f(z_1 z_2)$, where $f(u)$ is an arbitrary entire function of single complex variable.

Therefore, it generates the necessity to study partial entire solutions of these equations using natural formulations of problems for them.

The paper initiates a series of papers devoted index boundedness of entire solutions for the Cauchy and the boundary problems in the case of the second order partial differential equations. At the end, we pose open problems concerning conditions of the directional L-index boundedness for holomorphic solutions of the Cauchy problem of the heat equation. The conditions will allow a qualitative description of local and asymptotic behavior of the heat equation holomorphic solutions presenting the temperature distribution in the process of plasma electrolytic oxidation.

2. Main definitions

An entire function $F(z)$, $z \in \mathbb{C}^n$, is called (see [9]) *a function of bounded L-index in a direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{C}^n$

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le \max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| : 0 \le k \le m_0\right\}, \qquad (1)$$

where $\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} := F(z)$, $\frac{\partial F(z)}{\partial \mathbf{b}} := \sum\limits_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j = \langle \mathbf{grad} F, \overline{\mathbf{b}} \rangle$, $\frac{\partial^k F(z)}{\partial \mathbf{b}^k} := \frac{\partial}{\partial \mathbf{b}}\left(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}}\right)$, $k \ge 2$.

The least such integer $m_0 = m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *of the entire function* $F(z)$ *and is denoted by* $N_{\mathbf{b}}(F,L) = m_0$. If such m_0 does not exist then F is called *a function of unbounded L-index in the direction* $\mathbf{b}$ and we suppose that $N_{\mathbf{b}}(F,L) = \infty$. If $L(z) \equiv 1$ then $F(z)$ is called *a function of bounded index in the direction* $\mathbf{b}$ and $N_{\mathbf{b}}(F) = N_{\mathbf{b}}(F,1)$.

For $\eta > 0$, $z \in \mathbb{C}^n$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and a positive continuous function $L\colon\ \mathbb{C}^n \to \mathbb{R}_+$ we define

$$\lambda_1^{\mathbf{b}}(z,t_0,\eta) = \inf\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})} : |t-t_0| \le \frac{\eta}{L(z+t_0\mathbf{b})}\right\},$$

$$\lambda_1^{\mathbf{b}}(z,\eta) = \inf\{\lambda_1^{\mathbf{b}}(z,t_0,\eta) : t_0 \in \mathbb{C}\}, \ \ \lambda_1^{\mathbf{b}}(\eta) = \inf\{\lambda_1^{\mathbf{b}}(z,\eta) : z \in \mathbb{C}^n\},$$

$$\lambda_2^{\mathbf{b}}(z,t_0,\eta) = \sup\left\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})} : |t-t_0| \le \frac{\eta}{L(z+t_0\mathbf{b})}\right\},$$

$$\lambda_2^{\mathbf{b}}(z,\eta) = \sup\{\lambda_2^{\mathbf{b}}(z,t_0,\eta) : t_0 \in \mathbb{C}\}, \ \ \lambda_2^{\mathbf{b}}(\eta) = \sup\{\lambda_2^{\mathbf{b}}(z,\eta) : z \in \mathbb{C}^n\}.$$

By $Q_{\mathbf{b}}^n$ we denote the class of functions L which satisfy the condition

$$(\forall \eta \ge 0):\ 0 < \lambda_1^{\mathbf{b}}(\eta) \le \lambda_2^{\mathbf{b}}(\eta) < +\infty. \qquad (2)$$

It is known that a product of two entire functions of bounded L-index in direction is a function with the same class [5]. But the class of entire functions of bounded index is not closed under addition. The example was constructed by W. Pugh (see [11] and also [12]). Recently we generalized Pugh's example for entire functions of bounded L-index in direction [5].

Meanwhile, there are direct sufficient conditions of index boundedness for a sum of two entire functions [11], and implicit sufficent conditions for the sum by the soltuions of differential equations [13]. These results were generalized for multivariate entire functions [9, 7], analytic functions in the unit ball [8], slice entire functions [4], slice analytic functions in the unit ball [6].

We consider an arbitrary hyperplane $A = \{z \in \mathbb{C}^n : \langle z, c\rangle = 1\}$, where $\langle c, \mathbf{b}\rangle \neq 0$. Obviously that $\bigcup_{z^0\in A}\{z^0 + t\mathbf{b} : t \in \mathbb{C}\} = \mathbb{C}^n$.

Let $z^0 \in A$ be a given point. If $F(z^0 + t\mathbf{b}) \not\equiv 0$ as a function of variable $t \in \mathbb{C}$ then there exists $t_0 \in \mathbb{C}$ such that $F(z^0 + t_0\mathbf{b}) \neq 0$. Thus, for every line $\{z^0 + t\mathbf{b} : F(z^0 + t\mathbf{b}) \not\equiv 0\}$ we fixed one point t_0 with specified property. By B we denote a union of those points $z^0 + t_0\mathbf{b}$ i. e. $B = \bigcup\limits_{\substack{z^0\in A\\ F(z^0+t\mathbf{b})\not\equiv 0}} \{z^0 + t_0\mathbf{b}\}$.

Theorem 2.1 ([9]). *Let $L \in Q^n_{\mathbf{b}}$, $\alpha \in (0,1)$ and F, G be the entire in $\mathbb{C}^n$ functions satisfying conditions:*

1) *$G(z)$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$.*
2) *for every $z = z^0 + t\mathbf{b} \in \mathbb{C}^n$, where $z^0 \in A$, $z^0 + t_0\mathbf{b} \in B$ and $r = |t - t_0|L(z^0 + t\mathbf{b})$ the following inequality is valid*

$$\max\left\{|F(z^0 + t'\mathbf{b})| : \ |t' - t_0| = \frac{2r}{L(z^0 + t\mathbf{b})}\right\} \leq$$
$$\leq \max\left\{\frac{1}{k!L^k(z^0 + t\mathbf{b})}\left|\frac{\partial^k G(z^0 + t\mathbf{b})}{\partial \mathbf{b}^k}\right| : 0 \leq k \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\}.$$

3) $$c := \sup_{z^0+t_0\mathbf{b}\in B} \frac{\max\left\{|F(z^0 + t'\mathbf{b})| : \ |t' - t_0| = \frac{2\lambda_2^{\mathbf{b}}(1)}{L(z^0+t_0\mathbf{b})}\right\}}{|F(z^0 + t_0\mathbf{b})|} < \infty.$$

If $|\varepsilon| \leq \frac{1-\alpha}{2c}$ then the function $H(z) = G(z) + \varepsilon F(z)$ is of bounded L-index in the direction $\mathbf{b}$ with $N_{\mathbf{b}}(H, L) \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)$, where $G_\alpha(z) = G(z/\alpha)$, $L_\alpha(z) = L(z/\alpha)$.

3. Main theorem

The D'Alambert formula is the formula that describe solution of the Cauchy problem for the one-dimensional non-homogomogeneous wave equation:

$$\frac{\partial^2}{\partial x^2}u(x,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(x,t) = f(x,t). \tag{3}$$

in the domain $t > 0$, $-\infty < x < \infty$ with initial conditions

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x). \tag{4}$$

The formula has the following form

$$u(x,t)=\frac{\varphi(x+ct)+\varphi(x-ct)}{2}+\frac{1}{2c}\int\limits_{x-ct}^{x+ct}\psi(\alpha)d\alpha+\frac{1}{2c}\int\limits_{0}^{t}\int\limits_{x-c(t-\tau)}^{x+c(t-\tau)}f(s,\tau)dsd\tau. \tag{5}$$

We will consider the formula in two-dimensional complex space $\mathbb{C}^2$, i.e. $x\in\mathbb{C}$, $t\in\mathbb{C}$.

Since the D'Alambert formula contains three addendums we will apply Theorem 2.1 to deduce the following theorem.

Theorem 3.1. *Let $\varphi:\mathbb{C}\to\mathbb{C}$, $\psi:\mathbb{C}\to\mathbb{C}$ be entire functions, $l:\mathbb{C}\to\mathbb{R}_+$, $l\in Q$, $c\in\mathbb{C}\setminus\{0\}$. The following conditions are satisfied:*

(1) φ is of bounded l-index $N(\varphi,l)<\infty$

(2) the function $\mathfrak{F}(x,t)=\varphi(x+ct)+\varphi(x-ct)$ is of bounded $\mathfrak{L}$-index in the direction $\mathbf{b}=(b_1,b_2)$, where the function L is given by the formula $\mathfrak{L}(x,t)=\max\{l(x-ct),l(x+ct)\}$ and $b_1-b_2c\neq 0$;

(3) For every point $t\in\mathbb{C}$ the function $\Psi_t(\tau)=\int\limits_{(b_1-b_2c)\tau}^{2ct+(b_1+b_2c)\tau}\psi(\alpha)d\alpha\not\equiv 0$;

(4) For every point $t\in\mathbb{C}$ one has $\max\limits_{|t'-\tau_0|=2|t-\tau_0|}\Big|\int\limits_{(b_1-b_2c)t'}^{2ct+(b_1+b_2c)t'}\psi(s)ds\Big|\leq$

$$\leq c\max_{0\leq k\leq N_{\mathbf{b}}(\mathfrak{F}_\alpha,\mathfrak{L}_\alpha)}\Big\{|(b_1-cb_2)^k\cdot\varphi^{(k)}((b_1-b_2c)t)+$$

$$+(b_1+cb_2)^k\cdot\varphi^{(k)}((b_1+(b_2+2)c)t)|/(k!\mathfrak{L}^k((c_1+b_1)t,(1+b_2)t))\Big\}.$$

where τ_0 is chosen such that $\Psi_t(\tau_0)\neq 0$;

(5) There exist such positive constant $\mathfrak{C}$ such that

$$\max\Big\{\Big|\int\limits_{(b_1-b_2c)t'}^{2ct+(b_1+b_2c)t'}\psi(s)ds\Big|:|t'-\tau_0|=2\lambda_2^{\mathbf{b}}(1)/\mathfrak{L}(c_1t+b_1\tau_0,t+b_2\tau_0)\Big\}\leqslant$$

$$\leqslant\mathfrak{C}\Big|\int\limits_{(b_1-b_2c)\tau_0}^{2ct+(b_1+b_2c)\tau_0}\psi(\alpha)d\alpha\Big|. \tag{6}$$

If $|\varepsilon|<\frac{1}{2\mathfrak{C}}$ then the function $\mathfrak{H}(x,t)=\frac{\varphi(x+ct)+\varphi(x-ct)}{2}+\frac{\varepsilon}{2c}\int\limits_{x-ct}^{x+ct}\psi(\alpha)d\alpha$ is of bounded $\mathfrak{L}$-index in the direction $\mathbf{b}$ with $N_{\mathbf{b}}(\mathfrak{H},\mathfrak{L})\leq N_{\mathbf{b}}(\mathfrak{F},\mathfrak{L})$,

Proof. Denote $\mathbf{c}_- = (1, -c)$. Since φ is of bounded l-index,

$$\frac{\partial^m \varphi(x-ct)}{\partial \mathbf{b}^m} = \langle \mathbf{b}, \mathbf{c}_- \rangle^m \varphi^{(m)}(x-ct) \text{ for any } m \in \mathbb{N},$$

the function $\varphi(x-ct)$ is of bounded l_1-index in any direction $\mathbf{b} = (b_1, b_2) \neq (0,0)$, where $l_1(x,t) = l(x-ct)$. Similarly, the function $\varphi(x+ct)$ is of bounded l_2-index in any direction $\mathbf{b} = (b_1, b_2) \neq (0,0)$, where $l_2(x,t) = l(x+ct)$.

It is known [1] that if $l_1 \leqslant l_2$ for any and $f : \mathbb{C} \to \mathbb{C}$ has bounded l_1-index in a direction $\mathbf{b}$, then $f : \mathbb{C} \to \mathbb{C}$ is also of bounded l_2-index in the same direction $\mathbf{b}$. Given this fact, every function $\varphi(x-ct)$ and $\varphi(x+ct)$ is of bounded $\mathfrak{L}$-index in the direction $\mathbf{b} = (b_1, b_2)$.

The function $\mathfrak{L}(x,t) = \max\{l(x-ct), l(x+ct)\}$ belongs to the class $Q^2_{\mathbf{b}}$. This fact can be proved by a direct check of inequality $\lambda_2^{\mathbf{b}}(\eta) < \infty$ is finite because $l \in Q$ by Condition 1) (see similar statements and corresponding tehcniques for the classes $Q^n_{\mathbf{b}}$ in [14], Q^n in [15], $Q_{\mathbf{b}}(\mathbb{D}^n)$ in [16], respectively.

Condition 3) in the theorem provides existence of non-empty sets A and B in Theorem 2.1: $A = \{(z,t) \in \mathbb{C}^2 : z - ct = 0\} = \{(ct, t) : \ t \in \mathbb{C}\}$,

$$B = \bigcup_{t \in \mathbb{C}} \{(ct + b_1\tau, t + b_2\tau) : \int\limits_{(b_1 - b_2 c)\tau}^{2ct + (b_1 + b_2 c)\tau} \psi(\alpha) d\alpha \neq 0\}.$$

Condition 4) in the theorem gives validity of Condition 3) in Theorem 2.1 with $(z_1^0, z_2^0) = (ct, t)$, $t_0 = \tau_0$, $F(z_1, z_2) = \frac{1}{2c} \int_{z_1 - cz_2}^{z_1 + cz_2} \psi(s)\, ds$,

$$G(z_1, z_2) = \frac{\varphi(z_1 - cz_2) + \varphi(z_1 + cz_2)}{2}, \ L(z_1, z_2) = \mathfrak{L}(z_1, z_2).$$

Similarly, Condition 5) in the theorem yields validity of Condition 4) in Theorem 2.1. Thus, applying Theorem 2.1 we obtain the desired conclusion. □

Problem 1. Let $\varphi : \mathbb{C} \to \mathbb{C}$, $\psi : \mathbb{C} \to \mathbb{C}$ be entire functions, $c \in \mathbb{C} \setminus \{0\}$, φ is of bounded l-index $N(\varphi, l) < \infty$, and $\mathfrak{L}(x,t) = \max\{l(x-ct), l(x+ct)\}$.

Has always the function $\mathfrak{F}(x,t) = \varphi(x+ct) + \varphi(x-ct)$ the bounded $\mathfrak{L}$-index in the direction $\mathbf{b} = (b_1, b_2)$?

Let us consider the simplest Cauchy problem for the heat equation:

$$u'_t = a^2 u_{xx}, (x \in \mathbb{R}, \ t \in (0, +\infty)), \ u(x,t)\Big|_{t=0} = \varphi(x).$$

Its solution given by the Fourier transform method is the following

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \varphi(\lambda) \exp\left(-\frac{(x-\lambda)^2}{4a^2 t}\right) d\lambda, \qquad (7)$$

if the integral $\int_{-\infty}^{\infty} |f(x)| dx$ is converged.

Let φ be an infinitely differentiable function supported in $[-A, A]$. Then by the Paley-Wiener Theorem there exists an entire function of exponential type such that it is just ordinary Fourier transform of the function φ. Mroeover, every entire function of single variable with bounded index is a function of exponential type. This leads to the following question:

Problem 2. What are the function L and the direction $\mathbf{b}$ such that an analytic continuation of the function $u(x,t)$ given by (7) has bounded L-index in the direction $\mathbf{b}$?

A full answer to the question will allow to study local and asymptotic behavior of analytic solutuions for multidimensional heat equation in various geometric domains. In particular, it will be applicable to describe temperature distributuion in the cylindrical bodies whose coating is formed by the plasma electolytic oxidation.

Acknowledgments. The research was funded by the National Research Foundation of Ukraine (the project 2023.04/0160, 0124U003748 "Development of technology for the formation of heat-resistant coatings by plasma electrolytic oxidation on parts of aviation and rocket and space technics").

References

1. Bandura A., Skaskiv O. Entire functions of several variables of bounded index. Publisher I. E. Chyzhykov, Chyslo, Lviv (2016)
2. Goldberg, A. On the growth of entire solutions of algebraic differential equations. *Lith. Math. J.* 45 (1), 44–49 (2005). https://doi.org/10.1007/s10986-005-0005-8
3. Bandura A., Salo T., Skaskiv O. L-Index in Joint Variables: Sum and Composition of an Entire Function with a Function With a Vanished Gradient. *Fractal and Fractional 7* (8), 593 (2023). https://doi.org/10.3390/fractalfract7080593
4. Bandura A., Skaskiv O. Sum and product of functions having bounded L-index in a direction which are slice holomorphic in the same direction. *Prykarpatskyi Visn. Nauk. Tov. Im. Shevchenka. Chyslo no.1(53)*, 9–20 (2019). https://doi.org/10.31471/2304-7399-2019-1(53)-9-20
5. Bandura A.I. Product of two entire functions of bounded L-index in direction is a function with the same class. *Bukovyn. Mat. Zh. 4* (1-2), 8–12 (2016).

6. Baksa V.P., Bandura A.I., Salo T.M. Boundedness of the L-index in a direction of the sum and product of slice holomorphic functions in the unit ball. *Mat. Stud. 57* (2), 216–224 (2022).
7. Bandura A. Boundedness of **L**-index in joint variables for sum of entire functions, *Kragujevac J. Math. 46* (4), 595–603 (2022). https://doi.org/10.46793/KgJMat2204.595B
8. Bandura A. Composition, product and sum of analytic functions of bounded *L*-index in direction in the unit ball. *Mat. Stud. 50* (2), 115–134 (2018). http://doi.org/10.15330/ms.50.2.115-134
9. Bandura A.I. Sum of entire functions of bounded *L*-index in direction. *Mat. Stud. 45* (2), 149–158 (2016).
10. Bandura,A.I., Petrechko,N.V. Sum of entire functions of bounded index in joint variables. *Electr. J. Math. Anal. Appl. 6* (2), 60–67 (2018). https://doi.org/10.21608/ejmaa.2018.312547
11. Pugh W.J. Sums of functions of bounded index. *Proc. Amer. Math. Soc. 22*, 319–323 (1969). https,//doi.org/10.1090/S0002-9939-1969-0243067-9
12. M. Sheremeta, Analytic functions of bounded index, VNTL Publishers, Lviv, 1999.
13. Roy R., Shah S. M. Sums of functions of bounded index and ordinary differential equations. *Complex Variables, Theory and Application, 12* (1–4), 95–100 (1989). https://doi.org/10.1080/17476938908814358
14. Bandura A. I. Properties of positive continuous functions in $\mathbb{C}^n$. *Carpathian Math. Publ. 7* (2), 137–147 (2015). http://doi.org//10.15330/cmp.7.2.137-147
15. Bandura A.I. Remarks on some classes of positive continuous functions in $\mathbb{C}^n$. *Precarp. bull. Shevchenko Sci. Soc. Number.* № 1(59), 9–15 (2020). http://doi.org/10.31471/2304-7399-2020-1(59)-9-15
16. Bandura A.I., Kurliak P.O. On positive continuous functions defined in the unit polydisc. *Precarp. bull. Shevchenko Sci. Soc. Number.* №18(68), 9-17 (2023). http://doi.org/10.31471/2304-7399-2022-18(68)-9-17
17. L. Ropyak, T. Shihab, A. Velychkovych, et all. Design of a Two-Layer Al–Al2O3 Coating with an Oxide Layer Formed by the Plasma Electrolytic Oxidation of Al for the Corrosion and Wear Protections of Steel, *Progr. Phys. Metals 24* (2), 319–365 (2023).

Received by editorial office 4 November 2024

ПРО ФУНКЦІЇ, ПОВ'ЯЗАНІ З АНАЛІТИЧНИМИ РОЗВ'ЯЗКАМИ ЗАДАЧІ КОШІ ДЛЯ ХВИЛЬОВОГО РІВНЯННЯ ТА РІВНЯННЯ ТЕПЛОПРОВІДНОСТІ

А.І. Бандура

Івано-Франківський національний технічний університет нафти і газу; 76019, вул. Карпатська, 15, Івано-Франківськ, Україна;

e-mail: andriykopanytsia@gmail.com

Досліджуються властивості цілих розв'язків задачі Коші для одновимірного однорідного гіперболічного рівняння. Розглядаючи аналітичне продовження розв'язків, заданих формулою Д'Аламбера, ми знаходимо деякі умови, що забезпечують обмеженість L-індексу за напрямком для функцій, пов'язаних з цими розв'язками. Зокрема, для однорідного хвильового рівняння $c^2\frac{\partial^2}{\partial x^2}u(x,t)=\frac{\partial^2}{\partial t^2}u(x,t)$ *з початковими умовами* $u(x,0)=\varphi(x)$, $u_t(x,0)=\psi(x)$ *його розв'язок має вигляд*

$$u(x,t)=\frac{\varphi(x+ct)+\varphi(x-ct)}{2}+\frac{1}{2c}\int\limits_{x-ct}^{x+ct}\psi(\alpha)d\alpha.$$

Натомість вивчаються функції вигляду $\mathfrak{H}(x,t)=\frac{\varphi(x+ct)+\varphi(x-ct)}{2}+\frac{\mathfrak{C}}{2c}\int\limits_{x-ct}^{x+ct}\psi(\alpha)d\alpha$, *де* x, t, $c\in\mathbb{C}$, $\mathfrak{C}$ *— додатна стала, що визначається з деяких умов на функції* φ *та* ψ. *Наш основний результат дає достатні умови обмеженості* $\mathfrak{L}$*-індексу за напрямком* $\mathbf{b}$ *для таких функцій* $\mathfrak{H}$. *Його доведення використовує відомі достатні умови для суми цілих функцій. Насамкінець ставимо відкриті питання про умови обмеженості L-індексу за напрямком для аналітичних розв'язків задачі Коші для рівняння теплопровідності. Ці умови дадуть змогу якісно описати локальну та асимптотичну поведінку аналітичних розв'язків параболічного рівняння, що описують розподіл температури під час плазмового електролітичного оксидування.*

Ключові слова: *цілий розв'язок, задача Коші, одновимірне гіперболічне рівняння, сума функцій, обмежений L-індекс за напрямком, хвильове рівняння, рівняння теплопровідності, ПЕО.*