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ON THE SUM AND MAXIMAL TERM OF TAYLOR-DIRICHLET TYPE SERIES

A.Yu. Bodnarchuk, O.B. Skaskiv
Ivan Franko National University of Lviv;
79000, 1 Universytetska street, Lviv, Ukraine;
e-mail: 8andriy1111@gmail.com, olskask@gmail.com

The article deals with Taylor-Dirichlet type series of the form $F(x)=\sum_{k=0}^{+\infty}a_ke^{x\lambda_k+\tau(x)\beta_k}$, *where* $\Lambda=(\lambda_k)$ *and* $\beta=(\beta_k)$ *are some sequences of non-negative numbers, and* $\tau(x)$ *is a non-negative non-decreasing function,* $a_k\geq 0$ $(k\geq 0)$. *The class of such functions we denote* $\mathscr{T}\mathscr{D}(\Lambda,\beta,\tau)$. *The main statement of the paper is Theorem 2: Let the sequence* $(\lambda_n+\beta_n)$ *be increasing, a sequence* β *be non-decreasing and a positive function* τ *be such that* $\tau(x+h)-\tau(x)\geqslant h$ $(x>0,h>0)$. *If the condition* $\sum_{k=0}^{\infty}(\lambda_{k+1}+\beta_{k+1}-\lambda_k-\beta_k)^{-1}<+\infty$ *is fulfilled, then for every function* $F\in\mathscr{T}\mathscr{D}(\Lambda,\beta,\tau)$ *the asymptotic relation* $F(x)=(1+o(1))\mu(x,F)$ *holds as* $x\to+\infty$ *outside some set* $E\subset[0,+\infty)$ *of finite Lebesgue measure* ($\int_E dx<+\infty$), *where* $\mu(x,F)=\max\{a_ke^{\tau(x)\beta_k+x\lambda_k}\colon k\geq 0\}$. *Theorem 2 was proved earlier (1998) under the conditions of strict increasing of the sequences* Λ *and* β.

Key words: *Taylor-Dirichlet series; exceptional set; maximal term.*

1. Introduction

Let $\mathscr{T}\mathscr{D}(\Lambda,\beta,\tau)$ be the class of absolutely convergent for all $x\geq 0$ the Taylor-Dirichlet type series of the form

$$F(x)=\sum_{k=0}^{+\infty}a_ke^{x\lambda_k+\tau(x)\beta_k} \tag{1}$$

such that $a_k\geq 0$ $(k\in\mathbb{Z}_+)$; here $\Lambda=(\lambda_k)$ is some sequence of the non-negative numbers $\lambda_k\geq 0$ $(k\geq 0)$, and $\beta=\big(\beta_k\big)$ is also sequence of the non-negative numbers, such that $(\lambda_k,\beta_k)\neq(\lambda_j,\beta_j)$ for all $k\neq j$, i.e. are different two-dimensional vectors; $\tau\colon[0,+\infty)\to(0,+\infty)$ is continuously differentiable non-decreasing function. In the case of $\beta_k\equiv 0$, we obtain a Dirichlet series of the form

$$F_1(x) = \sum_{k=0}^{+\infty} a_k e^{x\lambda_k}, \tag{2}$$

that converges for all $x \geq 0$, and will write $F_1 \in \mathscr{D}(\Lambda) = \mathscr{T}\mathscr{D}(\Lambda, 0, \tau)$; $\mathscr{T}\mathscr{D} := \cup_\Lambda \cup_\beta \mathscr{T}\mathscr{D}(\Lambda, \beta, \tau)$, $\mathscr{D} := \cup_\Lambda \mathscr{D}(\Lambda)$. It is clear that $\mathscr{D} \subset \mathscr{T}\mathscr{D}$.

For any $x \geq 0$ and a Taylor-Dirichlet series $F \in \mathscr{T}\mathscr{D}(\Lambda, \beta, \tau)$ of form (1) we denote by

$$\mu(x,F) = \max\{a_k e^{\tau(x)\beta_k + x\lambda_k} \colon k \geq 0\}, \ \ \nu(x,F) = \sup\{k \colon a_k e^{\tau(x)\beta_k + x\lambda_k} = \mu(x,F)\}$$

the maximal term and central index of series (1), respectively; for a Dirichlet series $F_1 \in \mathscr{D}(\Lambda)$ of form (2) we have $\mu(x, F_1) = \max\{a_k e^{x\lambda_k} \colon k \geq 0\}$.

It easy to see that for each function $F \in \mathscr{T}\mathscr{D}(\Lambda, \beta, \tau)$ of form (1) there exists $\nu(x,F) = \max\{k \colon a_k e^{\tau(x)\beta_k + x\lambda_k} = \mu(x,F)\}$, i.e., in particular,

$$a_{\nu(x,F)} e^{\tau(x)\beta_{\nu(x,F)} + x\lambda_{\nu(x,F)}} = \mu(x,F).$$

2. Main results

The following lemma contains conditions on the sequences λ and β which are sufficient for the existence of the central index $\nu(x,F)$ for every function $F \in \mathscr{T}\mathscr{D}(\Lambda, \beta, \tau)$.

Lemma 2.1. *Let $F \in \mathscr{T}\mathscr{D}(\Lambda, \beta, \tau)$ and one of the following four conditions is fulfilled:*
i_1) a function $\tau(x)$ is non-decreasing, a sequence (β_n) is non-decreasing and a sequence (λ_n) is increasing;
i_2) a function $\tau(x)$ is increasing, a sequence (β_n) is also increasing and a sequence (λ_n) is nondecreasing;
ii_1) a function $\tau(x)$ is such that $\tau(x+h) - \tau(x) \leqslant h$ $(x > 0, h > 0)$, a sequence $\alpha = (\lambda_n + \beta_n)$ is an increasing sequence and a sequence $\lambda = (\lambda_n)$ is a non-decreasing sequence;
ii_2) a function $\tau(x)$ is such that $\tau(x+h) - \tau(x) \geqslant h$ $(x > 0, h > 0)$, a sequence $\alpha = (\lambda_n + \beta_n)$ is an increasing sequence and a sequence $\beta = (\beta_n)$ is a non-decreasing sequence.
Then the central index $\nu(x,F)$ is a non-decreasing function such that $\nu(x,F) < +\infty$ for every $x \geq 0$.

Proof of Lemma 2.1. Below, where it will not cause ambiguity, instead of the notations $\mu(x,F)$ and $\nu(x,F)$, we use the notation $\mu(x)$ and $\nu(x)$, respectively. Let $F \in \mathscr{T}\mathscr{D}(\Lambda, \beta, \tau)$, $x > 0$, $h \in (-x, +\infty)$. Since $\ln a_{\nu(x+h)} + x\lambda_{\nu(x+h)} +$

$\tau(x)\beta_{\nu(x+h)} \leq \ln\mu(x,F)$,

$$\ln\mu(x+h) = \ln a_{\nu(x+h)} + (x+h)\lambda_{\nu(x+h)} + \beta_{\nu(x+h)}\tau(x+h) =$$
$$= \ln a_{\nu(x+h)} + x\lambda_{\nu(x+h)} + \tau(x)\beta_{\nu(x+h)} + h\lambda_{\nu(x+h)} + \beta_{\nu(x+h)}(\tau(x+h)-\tau(x)) \leqslant$$
$$\leqslant \ln\mu(x) + h\lambda_{\nu(x+h)} + \beta_{\nu(x+h)}(\tau(x+h)-\tau(x)). \qquad (3)$$

Similarly,

$$\ln\mu(x+h) \geqslant \ln a_{\nu(x)} + (x+h)\lambda_{\nu(x)} + \beta_{\nu(x)}\tau(x+h) =$$
$$= \ln\mu(x) + h\lambda_{\nu(x)} + \beta_{\nu(x)}(\tau(x+h)-\tau(x)). \qquad (4)$$

From inequalies (3), (4), we obtain

$$h\lambda_{\nu(x)} + \beta_{\nu(x)}(\tau(x+h)-\tau(x)) \leqslant \ln\mu(x+h) - \ln\mu(x) \leqslant$$
$$\leqslant h\lambda_{\nu(x+h)} + \beta_{\nu(x+h)}(\tau(x+h)-\tau(x)) \qquad (5)$$

and $A(h) := h(\lambda_{\nu(x+h)} - \lambda_{\nu(x)}) + (\beta_{\nu(x+h)} - \beta_{\nu(x)})(\tau(x+h)-\tau(x)) \geqslant 0$ (6)

for all $x>0$ and $h>-x$. Hence, in case i_1), that the function $\tau(x)$ is non-decreasing, the sequence (β_n) is non-decreasing and the sequence (λ_n) is increasing, reasoning from opposite, we assume that $\nu(x) > \nu(x+h)$ for some x and $h>0$. Then $\tau(x+h)-\tau(x) \geq 0$, $\lambda_{\nu(x+h)} < \lambda_{\nu(x)}$, $\beta_{\nu(x+h)} \leq \beta_{\nu(x)}$, thus $A(h) < 0$ and we obtain a contadiction with (6).

Similarly, in case i_2), if the function $\tau(x)$ is increasing, the sequence (β_n) is also increasing and the sequence (λ_n) is non-decreasing we have that the cental index $\nu(x) = \nu(x,F)$ is the non-decreasing function. Indeed, reasoning from opposite, we assume that $\nu(x) > \nu(x+h)$ for some x and $h>0$, so $\tau(x+h)-\tau(x) > 0$, $\lambda_{\nu(x+h)} \leq \lambda_{\nu(x)}$, $\beta_{\nu(x+h)} < \beta_{\nu(x)}$, and therefore $A(h) < 0$ and we again get a contadiction with (6).

ii_1) If $\tau(x+h)-\tau(x) \leq h$ for all $x>0, h>0$, $\alpha = (\lambda_n + \beta_n)$ is an increasing sequence, then if we assume that there exist $x>0$ and $h>0$ such that $\nu(x+h) < \nu(x)$, then in the case $\beta_{\nu(x+h)} \geqslant \beta_{\nu(x)}$ in view of the inequality $\tau(x+h)-\tau(x) \leqslant h$ and increasing of the sequence $(\lambda_n+\beta_n)$, we get $0 \leqslant A(h) \leqslant h(\lambda_{\nu(x+h)} + \beta_{\nu(x+h)} - \lambda_{\nu(x)} - \beta_{\nu(x)}) < 0$, that it is impossible.

In the case $\beta_{\nu(x+h)} < \beta_{\nu(x)}$, in view of non-decreasing a sequence (λ_n), we directly also obtain contradiction with (6).

ii_2) This case it follows from ii_1) by using statement ii_1) for the function

$$F_1(x) = F(\tau^{-1}(x)) = \sum_{n=0}^{+\infty} a_n e^{x\beta_n + \lambda_n \tau^{-1}(x)}.$$

□

In article [1], it is proved the following theorem.

Theorem 2.1 (Skaskiv, Trusevych, 1998). *Let the sequences $\Lambda = (\lambda_n), \beta = (\beta_n)$ be increasing, and a differentiable function τ be such that $\tau'(x) \geqslant 1$ $(x > 0)$. For every function $F \in \mathscr{T}\mathscr{D}(\Lambda, \beta, \tau)$ the asymptotic relation $F(x) = (1+o(1))\mu(x,F)$ holds as $x \to +\infty$ outside some set $E \subset [0,+\infty)$ of finite Lebesgue measure ($\int_E dx < +\infty$), if and only if*

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_{n+1} + \beta_{n+1} - \lambda_n - \beta_n} < +\infty. \tag{7}$$

Analysis of the proof of article [1] shows that the conditions, the sequences $\lambda = (\lambda_n), \beta = (\beta_n)$ are increasing, of Theorem 2.1 can be replaced by the condition $ii_1)$ from Lemma 2.1. This condition is weaker in a general case.

We prove the following theorem.

Theorem 2.2. *Let a sequence $(\lambda_n + \beta_n)$ be increasing, a sequence $\beta = (\beta_n)$ be non-decreasing and a positive differentiable function τ be such that $\tau(x+h) - \tau(x) \geqslant h$ $(x > 0, h > 0)$. If the condition (7) is fulfilled, then for every function $F \in \mathscr{T}\mathscr{D}(\Lambda, \beta, \tau)$ the asymptotic relation*

$$F(x) = (1+o(1))\mu(x,F) \tag{8}$$

holds as $x \to +\infty$ outside some set $E \subset [0,+\infty)$ of finite Lebesgue measure ($\int_E dx < +\infty$).

The proof of the Theorem 2.2 repeats the proof of Theorem 2.1 almost verbatim and uses the following auxiliary statements.

We assume that $\sum_{k=0}^{+\infty} (\alpha_{k+1} - \alpha_k)^{-1} < +\infty$, where $\alpha_k = \lambda_k + \beta_k$, and denote

$$\delta_k = \max\{(j-l+1)^{-3/2} \sum_{m=l}^{j} \frac{1}{\alpha_{m+1} - \alpha_m} : 1 \le l \le k-1 \le j < +\infty\}.$$

Lemma 2.2 ([2], Lemmas 1, 3). *If $\sum_{k=0}^{+\infty} (\alpha_{k+1} - \alpha_k)^{-1} < +\infty$, where $\alpha_k = \lambda_k + \beta_k$, then there exists a sequence (c_k) such that $0 \le c_k \uparrow +\infty$ $(k \uparrow +\infty)$, $\varepsilon_k = c_k \delta_k \in (0, 1/2)$ $(k \ge 0)$, $\sum_{k=0}^{+\infty} \varepsilon_k < +\infty$, and $\sum_{k \ne \nu} \exp\{-\varepsilon_k |\alpha_k - \alpha_\nu|\} = o(1)$ $(\nu \to +\infty)$.*

Lemma 2.3 ([1], Lemma 2)**.** *Let $\nu(x)\colon [0,+\infty)\to\mathbb{Z}_+$ be any non-decreasing function. If (ε_k) is a non-negative given sequence, then there exists a set $E\subset(0,+\infty)$ such that for all $x\in[0,+\infty)\setminus E$ the equalities $\nu(x\pm\varepsilon_{\nu(x)})=\nu(x)$ are holds and for Lebesgue measure of the set E we have*

$$\text{meas}\,\big(E\cap[0,R]\big)=\int\limits_{E\cap[0,R]}dx\le 2(C+\sum_{k=0}^{\nu(R-0)}\varepsilon_k)\quad(R>0),\tag{9}$$

and the non-negative constant C depends only on function $\nu(x)$.

Proof of Theorem 2.2. Since from ii_2) of Lemma 2.1 the function $\nu(x,F)$ is non-decreasing, by using Lemma 2.3 for function $\nu(x)=\nu(x,F)$ we get the equalities $\nu(x\pm\varepsilon_{\nu(x)})=\nu(x)$ for all $x\in[0,+\infty)\setminus E$, and the set E satisfies (9).

Next, as in [1], we obtain by the definition of the maximal term of the series (1) and using by the equalities $\nu(x\pm\varepsilon_{\nu(x)})=\nu(x)$

$$a_ke^{(x\pm\varepsilon_{\nu(x)})\lambda_k+\tau(x\pm\varepsilon_{\nu(x)})\beta_k}\le\mu(x\pm\varepsilon_{\nu(x)},F)=\mu(x,F)e^{\pm\varepsilon_{\nu(x)}\lambda_{\nu(x)}+(\tau(x\pm\varepsilon_{\nu(x)})-\tau(x))\beta_{\nu(x)}},$$
$$\text{or } a_ke^{x\lambda_k+\tau(x)\beta_k}\leqslant\mu(x,F)e^{\pm\varepsilon_{\nu(x)}(\lambda_{\nu(x)}-\lambda_k)+(\tau(x\pm\varepsilon_{\nu(x)})-\tau(x))(\beta_{\nu(x)}-\beta_k)}$$

For $k<\nu(x)$ we have $(\tau(x-\varepsilon_{\nu(x)})-\tau(x))(\beta_{\nu(x)}-\beta_k)\le-\varepsilon_{\nu(x)}|\beta_{\nu(x)}-\beta_k|$, thus

$$a_ke^{x\lambda_k+\tau(x)\beta_k}\le\mu(x,F)\exp\{-\varepsilon_{\nu(x)}(\lambda_{\nu(x)}+\beta_{\nu(x,F)}-\lambda_k-\beta_k)\}$$

for all $k<\nu(x,F)$. For $k>\nu(x)$ we have $(\tau(x+\varepsilon_{\nu(x)})-\tau(x))(\beta_{\nu(x)}-\beta_k)\le\varepsilon_{\nu(x)}(\beta_{\nu(x)}-\beta_k)$, thus

$$a_ke^{x\lambda_k+\tau(x)\beta_k}\le\mu(x,F)\exp\{-\varepsilon_{\nu(x)}(\lambda_k+\beta_k-\lambda_{\nu(x)}-\beta_{\nu(x)})\}$$

for all $k>\nu(x,F)$. Finally, by Lemma 2.2 we obtain

$$F(x)/\mu(x,F)\le1+\sum_{k\ne\nu(x,F)}\exp\{-\varepsilon_{\nu(x,F)}|\lambda_{\nu(x)}+\beta_{\nu(x,F)}-\lambda_k-\beta_k|\}=$$
$$=1+\sum_{k\ne\nu(x,F)}\exp\{-\varepsilon_{\nu(x,F)}|\alpha_{\nu(x)}-\alpha_k|\}=1+o(1)$$

$x\to+\infty$ $(x\notin E)$. From Lemmas 2.2 and 2.3 we get

$$\text{meas}\,E\le2C+2\sum_{k=0}^{+\infty}\varepsilon_k<+\infty.$$

□

References

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ПРО СУМУ І МАКСИМАЛЬНИЙ ЧЛЕН РЯДУ ТИПУ ТЕЙЛОРА-ДІРІХЛЕ

А.Ю. Боднарчук, О.Б. Скасків
Львівський національний університет імені Івана Франка;
79000, вул. Університетська, 1, Львів, Україна;
e-mail: 8andriy1111@gmail.com, olskask@gmail.com

Розглядаються ряди типу Тейлора-Діріхле такого вигляду $F(x) = \sum_{k=0}^{+\infty} a_k e^{x\lambda_k+\tau(x)\beta_k}$*, де* $\Lambda = (\lambda_k)$ *і* $\beta = (\beta_k)$ *послідовності невід'ємних чисел,* $\tau(x)$ *невід'ємна неспадна функція,* $a_k \geq 0$ $(k \geq 0)$*. Клас таких функцій позначимо* $\mathscr{TD}(\Lambda,\beta,\tau)$*. Основними твердженнями статті є Теорема 2: Нехай послідовність* $(\lambda_n+\beta_n)$ *зростає, послідовність* β *неспадна, а додатна функція* τ *така, що* $\tau(x+h)-\tau(x) \geqslant h$ $(x>0, h>0)$*. Якщо виконується умова* $\sum_{k=0}^{\infty}(\lambda_{k+1}+\beta_{k+1}-\lambda_k-\beta_k)^{-1} < +\infty$*, то для кожної функції* $F \in \mathscr{TD}(\Lambda,\beta,\tau)$ *співвідношення* $F(x) = (1+o(1))\mu(x,F)$ *виконується при* $x \to +\infty$ *зовні деякої множини* $E \subset [0,+\infty)$ *скінченної міри Лебега (*$\int_E dx < +\infty$*), де* $\mu(x,F) = \max\{|a_k|e^{\tau(x)\beta_k+x\lambda_k} \colon k \geq 0\}$*. Теорему 2 було доведено раніше (1998) за умов строгого зростання послідовностей* Λ *і* β*.*

Ключові слова: *ряди Тейлора-Діріхле, виняткова множина; максимальний член.*