УДК 517.55

THE ORDER OF ENTIRE FUNCTIONS OF BOUNDED INDEX IN DIRECTION

A. I. Bandura, H. M. Kulinich, L. M. Shegda
Ivano-Frankivs'k national technical university of oil and gas;
76019, Ivano-Frankivs'k, Karpats'ka street, 15;
e-mail: bandura@nung.edu.ua

Estimates of order of entire solutions for partial differential equations are obtained. These solutions are functions of bounded index in direction.

Key words: *entire function, bounded L-index in direction, directional derivative, linear differential equation, order of entire function*

1. Introduction. B. S. Lee and S. M. Shah [3] investigated a type and order of entire solutions of linear homogeneous differential equtions with polynomial coefficients. We generalized their results for entire functions of bounded index in direction.

Let $L(z)$, $z \in \mathbb{C}^n$, be a positive continuous function.

Definition 1 *(see [5]). An entire function of $F(z)$, $z \in \mathbb{C}^n$, is called function of bounded L-index in the direction of $\mathbf{b} \in \mathbb{C}^n$, if there exists $m_0 \in \mathbb{Z}_+$ such that for $m \in \mathbb{Z}_+$ and every $z \in \mathbb{C}^n$ next inequality is true:*

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| : 0 \le k \le m_0\right\},$$

where $\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} = F(z)$, $\frac{\partial F(z)}{\partial \mathbf{b}} = \sum_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j$, $\frac{\partial^k F(z)}{\partial \mathbf{b}^k} = \frac{\partial}{\partial \mathbf{b}}\left(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}}\right), k \ge 2.$

The least such integer m_0 is called the L-index in direction of $F(z)$ and is denoted by $N_{\mathbf{b}}(F, L)$. If such m_0 does not exist then we put $N_{\mathbf{b}}(F, L) = \infty$ and F is said to be of unbounded L-index in direction. We also denote by $N_{\mathbf{b}}(F, L, z^0)$ as L-index in direction $\mathbf{b}$ of function F at a point z^0 that is the least integer m_0 for which inequality (1) is true at $z = z^0$. If $L(z) \equiv 1$ then F is called a function of bounded index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(F) \equiv N_{\mathbf{b}}(F, 1)$ is index in the direction $\mathbf{b}$.

Suppose that $W = F(z)$ is a transcendental entire function satisfying the linear partial differential equation

$$P_0(z)\frac{\partial^k W}{\partial \mathbf{b}^k}+P_1(z)\frac{\partial^{k-1} W}{\partial \mathbf{b}^{k-1}}+\cdots+P_k(z)W(z)=0,$$

where $P_j(z)$ are polynomials of degree not exceeding $\|d\|$,

$$P_{\|j\|}(z)=a_j z^d+\cdots,\ \ j,d\in \mathbb{Z}_+^n, \|j\|=0,1,\ldots,k,$$

and $a_0\neq 0$. Note that $a_j=0$ if the degree of $P_j(z)$ is less than $\|d\|$.

It is known [2] that if ρ denotes the order of $f(t),\ t\in\mathbb{C},$ then $\rho\le 1$, and $\rho=1$ if $a_k\neq 0$. Let us write

$$\tau=\tau[f]=\limsup_{r\to\infty}\frac{\log M(r,f)}{r},\ \gamma=\gamma[f]=\limsup_{r\to\infty}\frac{\nu(r,f)}{r}$$

where $\nu(r,f)=\nu(r)$ denotes the central index of $f(t)$, $M(r,f)=\max\{|f(t)|:|t|=r\}$. Then we have [1]

$$\tau\le\gamma\le N+1, \tag{3}$$

where N is an index of function $f(t)$. Let us denote $f_z(t)=F(z+t\mathbf{b})$, $\tau_z=\tau[f_z],\ \gamma_z=\gamma[f_z]$.

We call $N_\xi=N_{\mathbf{b}}(F,L,\xi)$ the local index in the direction $\mathbf{b}$ of $F(z)$ at $z=\xi$. It is easily seen that if $N_{\mathbf{b}}(F)<\infty$, then

$$N_{\mathbf{b}}(F)=\sup_{\xi}\{N_\xi\}.$$

The index set S_n of order n, $0\le n\le N_{\mathbf{b}}(F)$, is by definition the set of all points ξ such that $N_\xi=n$. Let

$$\sigma_n=\{r:|\xi|=r,\ \xi\in S_n\},\ 0\le n\le N_{\mathbf{b}}(F)$$

and let $m_l(\sigma_n)$ denote the logarithmic measure of $\sigma_n\cap[1,\infty)$.

2. Main result.

Theorem 1 *Let $F(z)$, $z\in\mathbb{C}^n$, be a transcendental entire function of bounded index. If $m_l(\sigma_p)=\infty$ for some $p>0$, then $\rho[f_z]=1$.*

We require the following three lemmas.

Lemma 1 [3] *Suppose that $g(r)$ is positive and non-decreasing for $r>r_0$. If a set Δ is contained in $1\le r<\infty$ and $m_l(\Delta)<\infty$, then*

$$\limsup_{\substack{r\to\infty\\ r\notin\Delta}}\frac{g(r)}{r}=\limsup_{r\to\infty}\frac{g(r)}{r}.$$

It is known [2] that if f is a transcendental entire function and $N_f<\infty$, then for any integer k (we choose $k\in\{0,1,\ldots,N(f)\}$, we have

$$f^{(j)}(t)=\left(\frac{\nu_k(r)}{t}\right)^{j-k} f^{(k)}(t)(1+\eta_j(t)),\nu_k(r)=\nu(r)(1+\delta_k(r)),k<j, \tag{4}$$

where $|\eta_j(t)| \to 0$ and $\delta_k(r) \to 0$ for $|t| = r \to \infty$ outside a set Δ_f of r of finite logarithmic measure. Here $\nu_k(r)$ is the central index of $f^{(k)}(t)$. We prove

Lemma 2 *Suppose* $N_{\mathbf{b}}(F) < \infty$ *with* $m_l(\sigma_p) = \infty$ *for some integer* p, $0 \le p \le N_{\mathbf{b}}(F)$. *Then for every* $z \in \mathbb{C}^n$

$$p \le \limsup_{\substack{r\to\infty \\ r\in\sigma_n - \Delta}} \frac{\nu(r, f_z)}{r} \le p+1$$

where $\Delta = \Delta_F$.

Proof. The proof of this Lemma is similar to proof of Lemma 2 in [3].

Lemma 3 *Suppose that there exists an integer* m *and a number* $R > 0$ *such that* $N_{\mathbf{b}}(F, \xi) \le m$ *for all* $|\xi| > R$. *Then for all* $z \in \mathbb{C}^n$

$$\gamma = \limsup_{r\to\infty} \frac{\nu(r, f_z)}{r} \le m+1.$$

Proof. It is known [4] that for any entire function $F \not\equiv 0$,

$$N_{\mathbf{b}}(F, \xi) \le P = P(R, f) < \infty$$

for all $|\xi| \le R$. So $N_{\mathbf{b}}(F) \le \max(m, P)$ and consequently there are only finitely many index sets. Let

$$S_{n_1}, S_{n_2}, \ldots, S_{n_p}, (n_i < n_{i+1}, i = 1, 2, \ldots, p-1)$$

be the sequence of all non-empty index sets such that

$$m_l(\sigma_{n_i}) = \infty, i = 1, 2, \ldots, p.$$

Let

$$\sigma = \bigcup_{i=1}^{p} \sigma_{n_i}.$$

Then the complementary set $\Sigma = \overline{\sigma}$ has a finite logarithmic measure. Lemma 2 implies that we have

$$\limsup_{\substack{r\to\infty \\ r\in\sigma_{n_i} - \Delta}} \frac{\nu(r, f_z)}{r} \le n_i + 1 \le n_p + 1 \le m+1,$$

where $m_l(\Delta) < \infty$. Hence

$$\limsup_{\substack{r\to\infty \\ r \quad \Sigma\cup\Delta}} \frac{\nu(r)}{r} \le m+1.$$

Since $m_l(\Sigma \cup \Delta) < \infty$, the lemma now follows from Lemma 1.

Proof of Theorem 1. By Lemma 2, $\limsup_{r\to\infty}\frac{\nu(r,f_z)}{r}>0$ and so $\rho[f_z]\geq 1$. Since F is a function of bounded index in the direction **b** we have $\rho[f_z]=1$.

References

1. S. M. Shah, Entire functions of bounded index. Proc. Amer. Math. Soc. 19, 1017-1022 (1968).
2. H. Wittich, Neuere Untersuchungen fiber eindeutige analytische Funktionen. Berlin 1955.
3. Boo Sang Lee, S. M. Shah, The type of an entire function satisfying a linear differential equation // Arch. Math. – 1969. – **20**, No 6. – P. 616-622.
4. Bandura A. I., Skaskiv O. B. Sufficient sets for boundedness L-index in direction for entire functions // Mat.Stud. – 2008. – **30**, No 2. – P. 177-182.
5. Bandura A. I., Skaskiv O. B. Entire functions of bounded L-index in direction // Math. Stud. – 2007. – **27**, No 1. – P. 30-52 (Ukrainian).

Стаття надійшла до редакційної колегії 16.12.2014 р.

*Рекомендовано до друку д.ф.-м.н., професором **Загороднюком А.В.**, д.ф.-м.н., професором **Скасківим О.Б.** (м. Львів)*

ПОРЯДОК ЦІЛИХ ФУНКЦІЙ ОБМЕЖЕНОГО ІНДЕКСУ ЗА НАПРЯМОМ

А. І. Бандура, Г. М. Кулініч, Л. М. Шегда

Івано-Франківський національний технічний університет нафти і газу; 76019, Івано-Франківськ, вул. Карпатська, 15; тел. +380 (342) 72-71-31; e-mail: bandura@nung.edu.ua

Отримано оцінку порядку цілих розв'язків рівнянь з частиними похідними у класі функцій обмеженого індексу за напрямом.

Ключові слова: *ціла функція, обмежений L-індекс за напрямом, похідна за напрямом, лінійне диференційне рівняння, порядок цілої функції.*