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ON POSITIVE CONTINUOS FUNCTIONS DEFINED IN THE UNIT POLYDISC

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In theory of holomorphic functions having bounded L-index in a direction $\mathbf{b}$ *an auxiliary class of positive continuous functions L is important to describe properties of the holomorphic functions by some inequalities and estimates containing the function L. This class is defined by local behavior of the function L. In the simplest one-dimensional case, the function should not vary locally too quickly, i.e.* $L(r+O(1/L(r))) = O(L(r))$ *for* $r = |z| \to +\infty$. *The paper is devoted an analog of this function class for the unit polydisc, i.e. for the Cartesian product of the unit discs. There is proved an equivalence of three different approaches to define the class. It is described by the local behavior on the slice* $z+t\mathbf{b}$ *for given* z *from the unit polydisc and for a fixed direction* $\mathbf{b}$, *where the complex variable* t *belongs to some disc with radius dependent on* $\mathbf{b}$ *and* z. *These estimates must be fulfilled uniformly in all* z. *There is indicated a possible explicit form of functions belonging to the class. The form is given as a product of arbitrary positive continuous function defined in the closed unit polydisc and the minimum of the expressions* $1/(1-|z_j|)$ *in all variables* z_j.

Key words: *positive continuous function; unit polydisc; boundary behavior; bounded L-index in direction.*

1. Main definitions and notations

The paper is some addendum to paper [1]. There was introduced a notion of L-index in a direction $\mathbf{b}\in\mathbb{C}^n\setminus\{0\}$ for functions analytic in the unit polydisc. An auxiliary class $Q_{\mathbf{b}}(\mathbb{D}^n)$ is used to deduce meaningful results for the analytic functions. Our goal is study properties of the class and describe partially function belonging to the class. The presented results are analogs of results obtained for n-dimensional complex space [3] and for the unit ball [2].

Let $\mathbf{0}=(0,\dots,0)$, $\mathbf{b}=(b_1,\dots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ be a given direction, $\mathbb{R}_+=(0,+\infty)$, $\mathbb{D}^n=\{z\in\mathbb{C}^n : |z_j|<1, j\in\{1,2,\dots,n\}\}$ be a unit polydisc, $L\colon\ \mathbb{D}^n\to\mathbb{R}_+$ be a continuous function such that for all $z=(z_1,z_2,\dots,z_n)\in\mathbb{D}^n$

$$L(z)>\beta\max_{1\leqslant j\leqslant n}\frac{|b_j|}{1-|z_j|},\ \beta=\mathrm{const}>1. \tag{1}$$

Remark 1.1. *Notice that if $\eta\in[0,\beta]$, $z\in\mathbb{D}^n$ and $|t|\le\frac{\eta}{L(z)}$ then $z+t\mathbf{b}\in\mathbb{D}^n$. Indeed, using* (1) *we have*

$$|z_j+tb_j|\le|z_j|+|tb_j|\le|z_j|+\frac{\eta|b_j|}{L(z)}<|z_j|+\frac{\beta|b_j|}{\beta\max_{1\leqslant s\leqslant n}\frac{|b_s|}{1-|z_s|}}\leqslant|z_j|+\frac{|b_j|}{\frac{|b_j|}{1-|z_j|}}=1.$$

Since for each $j\in\{1,\dots,n\}$ one has $|z_j+tb_j|<1$, the point $z+t\mathbf{b}$ is contained in the unit polydisc.

The positivity and continuity of the function L and condition (1) are weak to explore the behavior of analytic function of bounded L-index in direction. Below we impose an extra condition on behavior of the function L.

For a given point $z\in\mathbb{D}^n$ we denote $D_z=\{t\in\mathbb{C} : z+t\mathbf{b}\in\mathbb{D}^n\}$. In other words, $D_z=\{t\in\mathbb{C} : |t|<\min_{1\leqslant j\leqslant n}\frac{1-|z_j|}{|b_j|}\}$. Here if $b_j=0$ then we suppose $\frac{1-|z_j|}{|b_j|}=+\infty$. For $\eta\in[0,\beta]$, $z\in\mathbb{D}^n$, we define

$$\lambda_1^{\mathbf{b}}(z,\eta)=\inf\left\{\frac{L(z+t\mathbf{b})}{L(z)} : |t|\le\frac{\eta}{L(z)}\right\},\quad \lambda_1^{\mathbf{b}}(\eta)=\inf\{\lambda_1^{\mathbf{b}}(z,\eta,L) : z\in\mathbb{D}^n\},$$

$$\lambda_2^{\mathbf{b}}(z,\eta)=\sup\left\{\frac{L(z+t\mathbf{b})}{L(z)} : |t|\le\frac{\eta}{L(z)}\right\},\quad \lambda_2^{\mathbf{b}}(\eta)=\sup\{\lambda_2^{\mathbf{b}}(z,\eta,L) : z\in\mathbb{D}^n\}.$$

Denote

$$\lambda_{\mathbf{b}}(\eta)=\sup_{z\in\mathbb{D}^n}\sup_{t_1,t_2\in D_z}\left\{\frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})} : |t_1-t_2|\le\frac{\eta}{\min\{L(z+t_1\mathbf{b}),L(z+t_2\mathbf{b})\}}\right\}.$$

The notation $Q_{\mathbf{b}}(\mathbb{D}^n)$ stands for a class of positive continuous functions $L: \mathbb{D}^n \to \mathbb{R}_+$, satisfying (1) and

$$(\forall \eta \in [0,\beta]): \ \lambda_{\mathbf{b}}(\eta) < +\infty. \tag{2}$$

Actually it is enough to require validity of any inequality in (2) for one value $\eta \in (0,\beta]$ (for $\eta = 0$ the inequality is trivial). If $n = 1$ then $Q(\mathbb{D}) \equiv Q_{\mathbf{1}}(\mathbb{D}^1)$.

2. Properties of functions belonging to $Q_{\mathbf{b}}(\mathbb{D}^n)$

Proposition 2.1. *Let $L: \mathbb{D}^n \to \mathbb{R}_+$ be a positive continuous function satisfying condition* (2). *Then the following statements are equivalent:*

(1) $(\forall \eta \in [0,\beta]): \ \lambda_{\mathbf{b}}(\eta) < +\infty$;
(2) $(\forall \eta \in [0,\beta]): \ 0 < \lambda_1^{\mathbf{b}}(\eta) \le \lambda_2^{\mathbf{b}}(\eta) < +\infty$;
(3) $(\exists \eta \in (0,\beta]): \ 0 < \lambda_1^{\mathbf{b}}(\eta) \le \lambda_2^{\mathbf{b}}(\eta) < +\infty$.

Proof. The proof of this proposition is elementary and uses the definition of the $\lambda_{\mathbf{b}}(\eta)$, $\lambda_1^{\mathbf{b}}(\eta)$, $\lambda_2^{\mathbf{b}}(\eta)$.

1) $\Rightarrow$ 2). Indeed, for $\eta \in [0,\beta]$ one has $|t| \leqslant \frac{\eta}{L(z)} < \frac{\eta}{\beta \max_{1\leqslant j\leqslant n} \frac{|b_j|}{1-|z_j|}} \leqslant \frac{\eta}{\beta} \min_{1\leqslant j\leqslant n} \frac{1-|z_j|}{|b_j|} \leqslant \min_{1\leqslant j\leqslant n} \frac{1-|z_j|}{|b_j|}$, that is $t \in D_z$. Then

$$\begin{gathered}\lambda_2^{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{D}^n} \sup\{L(z+t\mathbf{b})/L(z) : |t| \le \eta/L(z)\} = \\ = \sup_{z\in\mathbb{D}^n} \sup\{L(z+t\mathbf{b})/L(z+0\mathbf{b}) : |t-0| \le \eta/L(z)\} \leqslant \\ \leqslant \sup_{z\in\mathbb{D}^n} \sup_{t_1,t_2\in D_z} \left\{ \frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})} : |t_1-t_2| \le \frac{\eta}{\min\{L(z+t_1\mathbf{b}), L(z+t_2\mathbf{b})\}} \right\} = \\ = \lambda_{\mathbf{b}}(\eta) < +\infty.\end{gathered}$$

Similarly, for $\lambda_1^{\mathbf{b}}(\eta)$ we obtain

$$\begin{gathered}\lambda_1^{\mathbf{b}}(\eta) = \inf_{z\in\mathbb{D}^n} \inf\{L(z+t\mathbf{b})/L(z) : |t| \le \eta/L(z)\} = \\ = \left(\sup_{z\in\mathbb{D}^n} \sup\{L(z+0\mathbf{b})/L(z+t\mathbf{b}) : |t-0| \le \eta/L(z)\} \right)^{-1} \leqslant \\ \leqslant \left(\sup_{z\in\mathbb{D}^n} \sup_{t_1,t_2\in D_z} \left\{ \frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})} : |t_1-t_2| \le \frac{\eta}{\min\{L(z+t_1\mathbf{b}), L(z+t_2\mathbf{b})\}} \right\} \right)^{-1} =\end{gathered}$$

$$= 1/\lambda_{\mathbf{b}}(\eta) < +\infty.$$

2) $\Rightarrow$ 3). It is obvious because 2) holds for all $\eta \in [0, \beta]$, and in 3) we require validity for some $\eta \in (0, \beta]$.

3) $\Rightarrow$ 1).

Assume that (2) holds for some $\eta_0 \in (0, \beta]$. In this case inequality (2) is true for all $\eta \leqslant \eta_0$, because $\lambda_1^{\mathbf{b}}(\eta)$ is a decreasing function and $\lambda_2^{\mathbf{b}}(\eta)$ is an increasing function.

Let $\eta \leqslant \eta_0$. Then $L(z+t\mathbf{b}) \leq \lambda_2(\eta)L(z)$ for each $|t| \leq \frac{\eta}{L(z)}$ and all $z \in \mathbb{D}^n$. Put $t = t_2 - t_1$, where $t_2, t_1 \in D_z$. Then $L(z - t_1\mathbf{b} + t_2\mathbf{b}) \leq \lambda_2(\eta)L(z)$ for all $|t_2 - t_1| \leq \frac{\eta}{L(z)}$ and every $z \in \mathbb{D}^n$. Substituting $z = z^0 + t_1\mathbf{b}$, we obtain $L(z^0 + t_2\mathbf{b}) \leq \lambda_2(\eta)L(z^0 + t_1\mathbf{b})$ for all $|t_2 - t_1| \leq \frac{\eta}{L(z^0+t_1\mathbf{b})}$ and every $z^0 \in \mathbb{D}^n$, i.e., when $L(z^0 + t_1\mathbf{b}) \leq L(z^0 + t_2\mathbf{b})$ we have $\frac{L(z^0+t_2\mathbf{b})}{L(z^0+t_1\mathbf{b})} \leq \lambda_2(\eta)$ as $|t_2 - t_1| \leq \frac{\eta}{L(z^0+t_1\mathbf{b})}$. And when $L(z^0 + t_1\mathbf{b}) > L(z^0 + t_2\mathbf{b})$ we have $\frac{L(z^0+t_2\mathbf{b})}{L(z^0+t_1\mathbf{b})} < 1 \leq \lambda_2(\eta)$ as $|t_2 - t_1| \leq \frac{\eta}{L(z^0+t_2\mathbf{b})}$. Therefore, $\frac{L(z^0+t_2\mathbf{b})}{L(z^0+t_1\mathbf{b})} \leq \lambda_2(\eta)$ as $|t_2 - t_1| \leq \frac{\eta}{\min\{L(z^0+t_1\mathbf{b}), L(z^0+t_2\mathbf{b})\}}$, i.e., (2) is fulfilled.

Choose $\eta \in (\eta_0, \beta]$. In view of monotonicity $\lambda_{\mathbf{b}}$, $\lambda_1^{\mathbf{b}}$, $\lambda_2^{\mathbf{b}}$, we need to prove (2) for $\eta = \beta$.

Then

$$\lambda_2^{\mathbf{b}}(\beta) = \sup_{z \in \mathbb{D}^n} \sup\{L(z+t\mathbf{b})/L(z) : |t| \leq \beta/L(z)\} =$$

$$= \sup_{z \in \mathbb{D}^n} \sup \left\{ \frac{\frac{\eta_0}{\beta} L(z+t\mathbf{b})}{\frac{\eta_0}{\beta} L(z)} : |t| \leq \frac{\eta_0}{\frac{\eta_0}{\beta} L(z)} \right\} = \lambda_2^{\mathbf{b}}(\eta_0) < +\infty,$$

because the constant multiplier $\frac{\eta_0}{\beta}$ does not change property (2). If it holds for the function $L(z)$, then it holds for the function $\frac{\eta_0}{\beta} L(z)$. Similarly, $\lambda_1^{\mathbf{b}}(\beta) > 0$.

As above above for $\eta \leqslant \eta_0$, we repeat considerations and again deduce that $\lambda_{\mathbf{b}}(\eta) < \infty$ for all $\eta \in [0, \beta]$. $\square$

The following lemma suggests possible approach to compose a function belonging to $Q_{\mathbf{b}}(\mathbb{D}^n)$.

Lemma 2.1. *Let* $\overline{\mathbb{D}}^n = \{z \in \mathbb{C}^n : |z_j| \leqslant 1, j \in \{1, 2, \ldots, n\}\}$, $L : \overline{\mathbb{D}}^n \to \mathbb{R}_+$ *be a continuous function,* $m = \min\{L(z) : z \in \overline{\mathbb{D}}^n\}$. *Then* $\widetilde{L}(z) = \frac{\beta_1}{m} \cdot L(z) \cdot \max_{1 \leqslant j \leqslant n} \frac{|b_j|}{(1-|z_j|)^\alpha} \in Q_{\mathbf{b}}(\mathbb{D}^n)$ *for every* $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$, $\alpha \geq 1$, $\beta_1 > \beta$.

Proof. We will check condition (1) for all $z \in \mathbb{D}^n$:

$$\widetilde{L}(z) = \frac{\beta_1}{m} \cdot L(z) \cdot \max_{1\leqslant j\leqslant n} \frac{|b_j|}{(1-|z_j|)^\alpha} >$$
$$> \frac{\beta}{m} \cdot \min\{L(z) : z \in \overline{\mathbb{D}}^n\} \cdot \max_{1\leqslant j\leqslant n} \frac{|b_j|}{(1-|z_j|)^\alpha} \geqslant \beta \max_{1\leqslant j\leqslant n} \frac{|b_j|}{(1-|z_j|)},$$

Using the definition of $Q_{\mathbf{b}}(\mathbb{D}^n)$ we have $\forall z \in \mathbb{D}^n$

$$\lambda_1^{\mathbf{b}}(z,\eta,\widetilde{L}) = \inf\left\{ \max_{1\leqslant j\leqslant n} \frac{|b_j|L(z+t\mathbf{b})}{(1-|z_j+tb_j|)^\alpha} \min_{1\leqslant j\leqslant n} \frac{(1-|z_j|)^\alpha}{|b_j|L(z)} : \right.$$
$$\left. |t| \leq \frac{\eta m}{\beta_1} \min_{1\leqslant j\leqslant n} \frac{(1-|z_j|)^\alpha}{|b_j|L(z)} \right\} \geq$$
$$\geq \inf\left\{ \frac{L(z+t\mathbf{b})}{L(z)} : |t| \leq \frac{\eta m}{\beta L(z)} \min_{1\leqslant j\leqslant n} \frac{(1-|z_j|)^\alpha}{|b_j|} \right\} \times$$
$$\inf\left\{ \frac{\min_{1\leqslant j\leqslant n}(1-|z_j|)^\alpha/|b_j|}{\min_{1\leqslant j\leqslant n}(1-|z_j+tb_j|)^\alpha/|b_j|)} : |t| \leq \frac{\eta m}{\beta L(z)} \min_{1\leqslant j\leqslant n} \frac{(1-|z_j|)^\alpha}{|b_j|} \right\}$$

In view of Remark 1.1 the points $z+t\mathbf{b}$ and z belong to the unit polydisc $\mathbb{D}^n$. But the function $L(z)$ is positive continuous in $\overline{\mathbb{D}}^n$. Thus, it is bounded and non-vanished in $\overline{\mathbb{D}}^n$. Thus, the first infimum is not lesser than a some constant $K > 0$ which is independent from z. Besides, we have $\forall z \in \mathbb{D}^n$ and $\forall t \in D_z$ $\frac{m}{L(z)} \leq 1$. Thus, for the second infimum the following estimates are valid

$$\inf\left\{ \frac{\min_{1\leqslant j\leqslant n}(1-|z_j|)^\alpha/|b_j|}{\min_{1\leqslant j\leqslant n}(1-|z_j+tb_j|)^\alpha/|b_j|} : |t| \leq \frac{\eta m}{\beta L(z)} \min_{1\leqslant j\leqslant n} \frac{(1-|z_j|)^\alpha}{|b_j|} \right\} \geq$$
$$\geq \inf\left\{ \frac{\min_{1\leqslant j\leqslant n}(1-|z_j|)^\alpha/|b_j|}{\min_{1\leqslant j\leqslant n}(1-|z_j+tb_j|)^\alpha/|b_j|} : |t| \leq \frac{\eta}{\beta} \min_{1\leqslant j\leqslant n} \frac{(1-|z_j|)^\alpha}{|b_j|} \right\} \geqslant$$
$$\geqslant \inf\left\{ \frac{\min_{1\leqslant j\leqslant n}(1-|z_j|)^\alpha}{\min_{1\leqslant j\leqslant n}(1-|z_j+tb_j|)^\alpha} : |t| \leq \frac{\eta}{\beta} \min_{1\leqslant j\leqslant n} \frac{(1-|z_j|)}{|b_j|} \right\} \cdot \frac{\min_{1\leqslant j\leqslant n}|b_j|}{\max_{1\leqslant j\leqslant n}|b_j|} =$$
$$= \left(\frac{1-|z_m|}{1-|z_s+t^*b_s|} \right)^\alpha \cdot \frac{\min_{1\leqslant j\leqslant n}|b_j|}{\max_{1\leqslant j\leqslant n}|b_j|}.$$

where $|t^*| \leq \frac{\eta}{\beta} \min_{1\leqslant j\leqslant n} \frac{(1-|z_j|)}{|b_j|}$, m, $s \in \{1,\ldots,n\}$. Now we find lower estimate for this fraction

$$\frac{1-|z_m|}{1-|z_s+t^*b_s|} \geq \frac{1-|z_m|}{1-||z_s|-|t^*b_s||} \geq \frac{1-|z_m|}{1-||z_s|-\frac{\eta|b_s|}{\beta}\min_{1\leqslant j\leqslant n}\frac{(1-|z_j|)}{|b_j|}|}.$$

Denoting $u_j = |z_j| \in [0;1)$, $\gamma = \frac{\eta}{\beta} \in [0,1]$, we consider a function of n real variables

$$s(u) = \frac{1-u_m}{1-|u_s-\gamma|b_s|\min_{1\leqslant j\leqslant n}\frac{1-u_j}{|b_j|}|}.$$

As in [2] it can be proved that the function $s(u)$ is greater than $\frac{1}{1+\gamma\frac{|b_s|}{\max_{1\leqslant j\leqslant n}|b_j|}}$.

In fact, we proved that

$$\lambda_1^{\mathbf{b}}(z,\eta,\widetilde{L}) \geq \frac{\min_{1\leqslant j\leqslant n}|b_j|}{\max_{1\leqslant j\leqslant n}|b_j|}\cdot\frac{1}{1+\frac{\eta}{\beta}\frac{|b_s|}{\max_{1\leqslant j\leqslant n}|b_j|}} > 0.$$

Hence, we have $\lambda_1^{\mathbf{b}}(\eta,\widetilde{L}) > 0$. By analogy, it can be proved that $\lambda_2^{\mathbf{b}}(\eta,\widetilde{L}) < \infty$. □

We often use the following properties $Q_{\mathbf{b}}(\mathbb{D}^n)$.

Lemma 2.2. *(1) If $L \in Q_{\mathbf{b},\beta}(\mathbb{D}^n)$ then for every $\theta \in \mathbb{C}\backslash\{0\}$ one has $L \in Q_{\theta\mathbf{b},\beta/|\theta|}(\mathbb{D}^n)$ and $|\theta|L \in Q_{\theta\mathbf{b},\beta}(\mathbb{D}^n)$*
(2) If $L \in Q_{\mathbf{b}_1,\beta}(\mathbb{D}^n)\bigcap Q_{\mathbf{b}_2,\beta}(\mathbb{D}^n)$ and for all $z \in \mathbb{D}^n$ one has

$$L(z) > \beta\max_{1\leqslant j\leqslant n}\frac{\max\{|b_{1,j}|,|b_{2,j}|,|b_{1,j}+b_{2,j}|\}}{1-|z_j|},$$

then $\min\{\lambda_2^{\mathbf{b}_1}(\beta,L),\lambda_2^{\mathbf{b}_2}(\beta,L)\}L \in Q_{\mathbf{b}_1+\mathbf{b}_2,\beta}(\mathbb{D}^n)$.

Proof. **1.** First, we prove that $(\forall\theta \in \mathbb{C}\backslash\{0\}) : L \in Q_{\theta\mathbf{b},\beta}(\mathbb{D}^n)$. Indeed, we have by definition

$$\lambda_1^{\theta\mathbf{b}}(z,\eta,L) = \inf\left\{\frac{L(z+t\theta\mathbf{b})}{L(z)} : |t| \leq \frac{\eta}{L(z)}\right\} =$$
$$= \inf\left\{\frac{L(z+(t\theta)\mathbf{b})}{L(z)} : |\theta t| \leq \frac{|\theta|\eta}{L(z)}\right\} = \lambda_1^{\mathbf{b}}(z,|\theta|\eta,L).$$

Therefore, we get

$$\lambda_1^{\theta\mathbf{b}}(\eta,L) = \inf\{\lambda_1^{\theta\mathbf{b}}(z,\eta,L) : z \in \mathbb{D}^n\} =$$
$$= \inf\{\lambda_1^{\mathbf{b}}(z,|\theta|\eta,L) : z \in \mathbb{D}^n\} = \lambda_1^{\mathbf{b}}(|\theta|\eta,L) > 0,$$

because $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$. Similarly, we prove that $\lambda_2^{\theta\mathbf{b}}(\eta,L) < +\infty$. But $|\theta|\eta \in [0,\beta]$. So $\eta \in [0,\beta/|\theta|]$. Thus, $L \in Q_{\theta\mathbf{b},\beta/|\theta|}(\mathbb{D}^n)$.

Let $L^* = |\theta| \cdot L$. Using definition of $\lambda_1^{\mathbf{b}}(z,\eta,L^*)$ we have

$$\lambda_1^{\theta\mathbf{b}}(z,\eta,L^*) = \inf\left\{\frac{L^*(z+t\theta\mathbf{b})}{L^*(z)} : |t| \le \frac{\eta}{L^*(z)}\right\} =$$
$$= \inf\left\{\frac{|\theta|L(z+t\theta\mathbf{b})}{|\theta|L(z)} : |t| \le \frac{\eta}{|\theta|L(z)}\right\} =$$
$$= \inf\left\{\frac{L(z+(t\theta)\mathbf{b})}{L(z)} : |\theta t| \le \frac{\eta}{L(z)}\right\} = \lambda_1^{\mathbf{b}}(z,\eta,L).$$

Therefore, we obtain

$$\lambda_1^{\theta\mathbf{b}}(\eta,L^*) = \inf\{\lambda_1^{\theta\mathbf{b}}(z,\eta,L^*) : z \in \mathbb{D}^n\} =$$
$$= \inf\{\lambda_1^{\mathbf{b}}(z,\eta,L) : z \in \mathbb{D}^n\} = \lambda_1^{\mathbf{b}}(\eta,L) > 0,$$

because $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$. Similarly, we prove that $\lambda_2^{\theta\mathbf{b}}(\eta,L^*) = \lambda_2^{\mathbf{b}}(\eta,L) < +\infty$. Thus, $L^* = |\theta| \cdot L \in Q_{\theta\mathbf{b},\beta}(\mathbb{D}^n)$.

2. It remains to prove a second part.

In view of Remark 1.1, if $z^0 \in \mathbb{D}^n$ and $|t| \le \frac{\eta}{L(z^0)}$ then $z^0 + t\mathbf{b}_1 \in \mathbb{D}^n$ and $z^0 + t\mathbf{b}_2 \in \mathbb{D}^n$.

Denote $L^*(z) = \min\{\lambda_2^{\mathbf{b}_1}(\beta,L), \lambda_2^{\mathbf{b}_2}(\beta,L)\} \cdot L(z)$. Assume that

$$\min\{\lambda_2^{\mathbf{b}_1}(\beta,L), \lambda_2^{\mathbf{b}_2}(\beta,L)\} = \lambda_2^{\mathbf{b}_2}(\beta,L).$$

Using definitions of $\lambda_1^{\mathbf{b}}(\eta,L)$, $\lambda_2^{\mathbf{b}}(\eta,L)$ and $Q_{\mathbf{b}}(\mathbb{D}^n)$ we obtain that

$$\inf\left\{\frac{L^*(z^0+t(\mathbf{b}_1+\mathbf{b}_2))}{L^*(z^0)} :\ |t| \le \frac{\eta}{L^*(z^0))}\right\} \ge$$
$$\ge \inf\left\{\frac{L^*(z^0+t\mathbf{b}_1+t\mathbf{b}_2)}{L^*(z^0+t\mathbf{b}_2)} : |t| \le \frac{\eta}{L^*(z^0)}\right\} \times$$
$$\times \inf\left\{\frac{L^*(z^0+t\mathbf{b}_2)}{L^*(z^0)} : |t| \le \frac{\eta}{L^*(z^0)}\right\} =$$
$$= \inf\left\{\frac{\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0+t\mathbf{b}_1+t\mathbf{b}_2)}{\lambda_2^{\mathbf{b}_1}(\beta,L)L(z^0+t\mathbf{b}_2)} : \eta/(\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0))\right\} \times$$
$$\times \inf\{\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0+t\mathbf{b}_2)/\lambda_2^{\mathbf{b}_1}(\beta,L)L(z^0) : |t| \le \eta/\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0)\} =$$
$$= \inf\{L(z^0+t\mathbf{b}_1+t\mathbf{b}_2)/L(z^0+t\mathbf{b}_2) : |t-t_0| \le \eta/(\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0+\mathbf{b}_2))\} \times$$
$$\times \inf\{L(z^0+t\mathbf{b}_2)/L(z^0) : |t| \le \eta/(\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0))\} \ge$$
$$\ge \inf\{L(z^0+t\mathbf{b}_1+t\mathbf{b}_2)/L(z^0+t\mathbf{b}_2) : |t| \le \eta/\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0)\} \times$$

$$\times \inf\left\{\frac{L(z^0+t\mathbf{b}_2)}{L(z^0)} : |t-t_0| \le \frac{\eta}{L(z^0)}\right\} \ge$$

$$\ge \inf\left\{\frac{L(z^0+t\mathbf{b}_1+t\mathbf{b}_2)}{L(z^0+t\mathbf{b}_2)} : |t-t_0| \le \frac{\eta}{(\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0))}\right\}\lambda_1^{\mathbf{b}_2}(z^0,\eta,L) \ge$$

$$\geqslant \lambda_1^{\mathbf{b}_2}(\eta,L)\frac{L(z^0+\hat{t}\mathbf{b}_1+\hat{t}\mathbf{b}_2)}{L(z^0+\hat{t}\mathbf{b}_2)} \qquad (3)$$

where $\hat{t}$ is a point at which infimum is attained

$$\frac{L(z^0+\hat{t}\mathbf{b}_1+\hat{t}\mathbf{b}_2)}{L(z^0+\hat{t}\mathbf{b}_2)} = \inf\left\{\frac{L(z^0+t\mathbf{b}_1+t\mathbf{b}_2)}{L(z^0+t\mathbf{b}_2)} : |t| \le \frac{\eta}{\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0)}\right\}.$$

But $L \in Q_{\mathbf{b}_2,\beta}(\mathbb{D}^n)$, then for all $\eta \in [0,\beta]$

$$\sup\left\{\frac{L(z^0+t\mathbf{b}_2)}{L(z^0)} : |t| \le \frac{\eta}{L(z^0)}\right\} \le \lambda_2^{\mathbf{b}_2}(\eta,L) < \infty.$$

Hence, $L(z^0+t\mathbf{b}_2) \le \lambda_2^{\mathbf{b}_2}(\eta,L)\cdot L(z^0)$, i.e. for $t=\hat{t}$ we have $L(z^0) \ge \frac{L(z^0+\hat{t}\mathbf{b}_2)}{\lambda_2^{\mathbf{b}_2}(\eta,L)}$. Using a proved inequality and (3), we obtain

$$\inf\{L^*(z^0+t(\mathbf{b}_1+\mathbf{b}_2))/L^*(z^0) : |t| \le \eta/L^*(z^0)\} \ge$$

$$\ge \lambda_1^{\mathbf{b}_2}(\eta,L)\inf\left\{L(z^0+t\mathbf{b}_1+\hat{t}\mathbf{b}_2)/L(z^0+\hat{t}\mathbf{b}_2) : |t| \le \frac{\eta}{\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0)}\right\} \ge$$

$$\ge \lambda_1^{\mathbf{b}_2}(\eta,L)\inf\left\{\frac{L(z^0+t\mathbf{b}_1+\hat{t}\mathbf{b}_2)}{L(z^0+\hat{t}\mathbf{b}_2)} : |t| \le \frac{\eta\lambda_2^{\mathbf{b}_2}(\eta,L)}{\lambda_2^{\mathbf{b}_2}(\beta,L)L(z^0!+\hat{t}\mathbf{b}_2)}\right\} \ge$$

$$\ge \lambda_1^{\mathbf{b}_2}(\eta,L)\cdot\inf\{L(z^0+t\mathbf{b}_1+\hat{t}\mathbf{b}_2)/L(z^0+\hat{t}\mathbf{b}_2) : |t| \le \eta/L(z^0+\hat{t}\mathbf{b}_2)\} =$$

$$= \lambda_1^{\mathbf{b}_2}(\eta,L)\lambda_1^{\mathbf{b}_1}(z^0+\hat{t}\mathbf{b}_2,\eta,L) \ge \lambda_1^{\mathbf{b}_2}(\eta,L)\lambda_1^{\mathbf{b}_1}(\eta,L).$$

Therefore, $\lambda_1^{\mathbf{b}_1+\mathbf{b}_2}(\eta,L^*) \ge \lambda_1^{\mathbf{b}_2}(\eta,L)\lambda_1^{\mathbf{b}_1}(\eta,L) > 0$. By analogy, we can prove that for all $\eta \in [0,\beta]$ one has $\lambda_2^{\mathbf{b}_1+\mathbf{b}_2}(\eta,L^*) < +\infty$. Thus, the function L^* belongs to the class $Q_{\mathbf{b}_1+\mathbf{b}_2,\beta}(\mathbb{D}^n)$. □

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ПРО ДОДАТНІ НЕПЕРЕРВНІ ФУНКЦІЇ, ВИЗНАЧЕНІ В ОДИНИЧНОМУ ПОЛІКРУЗІ

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У теорії голоморфних функцій обмеженого L-індексу за напрямком $\mathbf{b}$ допоміжний клас додатних неперервних функцій L є важливим для опису властивостей голоморфних функцій через деякі нерівності та оцінки, що містять функцію L. Цей клас визначається локальним поводженням функції L. У найпростішому одновимірному випадку функція з цього класу не повинна локально змінюватися надто швидко, тобто $L(r+O(1/L(r))) = O(L(r))$ для $r=|z| \to +\infty$. Стаття присвячена аналогу цього класу функцій для одиничного полікруга, тобто для декартового добутку одиничних дисків. Доведено еквівалентність трьох різних підходів до означення класу. Він описується локальною поводженням на зрізі $z+t\mathbf{b}$ для заданого z з одиничного полікруга та фіксованого напрямку $\mathbf{b}$, де комплексна змінна t міститься в деякому крузі з радіусом, залежним від $\mathbf{b}$ та z. Ці оцінки мають виконуватись рівномірно по всіх z. Вказано можливий явний вигляд функцій, що належать до цього класу. А саме деякі з них можна задати як добуток довільної додатної неперервної функції, визначеної на замкненому одиничному полікрузі, та мінімуму виразів $1/(1-|z_j|)$ по всіх змінних z_j.

Ключові слова: *додатна неперервна функція, одиничний полікруг, крайове поводження, обмежений L-індекс за напрямком.*