УДК 517.537.72+517.548.9
DOI: 10.31471/2304-7399-2023-18(68)-25-31

ON THE DOMAIN OF THE CONVERGENCE OF TAYLOR-DIRICHLET SERIES WITH COMPLEX EXPONENTS

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The article deals with Dirichlet series of the form $F(z)=\sum_{k=0}^{+\infty}a_ke^{z\lambda_k}$, *as well as Taylor-Dirichlet series of the form* $F_1(z)=\sum\limits_{k=0}^{+\infty}z^{m_k}a_ke^{z\lambda_k}$, *where* (λ_k) *is some sequence of pairwise different complex numbers, and* (m_k) *is a sequence of non-negative integers,* $a_k\in\mathbb{C}$. *The main statements of the work are Statements 1–3 and relate to the description and relationship between the domains of convergence, absolute convergence, and the domains of existence of the maximum term of the above series* F *and* F_1.

Key words: *Taylor-Dirichlet series; abscissa of the existence of maximal term.*

1. Introduction

Let $\mathscr{T}\mathscr{D}(\Lambda,\mathbf{m})$ be the class of formal Taylor-Dirichlet series of the form

$$F(z)=\sum_{k=0}^{+\infty}a_kz^{m_k}e^{z\lambda_k} \tag{1}$$

such that $a_k\to0$ $(k\to+\infty)$; here $\Lambda=(\lambda_k)$ is some sequence of the non-negative numbers $\lambda_k\geq0$ $(k\geq0)$, and $\mathbf{m}=(m_k)$ is a sequences of the non-negative integer numbers, such that $(\lambda_k,m_k)\neq(\lambda_j,m_j)$ for all $k\neq j$, i.e. are different two-dimensional vectors. In the case $m_k\equiv0$ we obtain a formal Dirichlet series

$$F_1(z)=\sum_{k=0}^{+\infty}a_ke^{z\lambda_k} \tag{2}$$

and will write $F_1\in\mathscr{D}(\Lambda)=\mathscr{T}\mathscr{D}(\Lambda,0)$; $\mathscr{T}\mathscr{D}:=\cup_\Lambda\cup_{\mathbf{m}}\mathscr{T}\mathscr{D}(\Lambda,\mathbf{m})$, $\mathscr{D}:=\cup_\Lambda\mathscr{D}(\Lambda)$. It is clear that $\mathscr{D}\subset\mathscr{T}\mathscr{D}$.

The convergence sets of Dirichlet series $F_1 \in \mathscr{D}(\Lambda)$ of form (2) with the complex exponents Λ have been studied by many authors. For example, such series were studied in the articles of W. Schnee [1, 2, 3], G.H. Hardy & M. Riesz [4], J.F. Ritt [7], E. Hille [8], J. Micusiński [9], T.M. Gallie [10, 11], G. Peyser [13], M.R. Kuryliak, O.B. Skaskiv [15] etc. J.F. Ritt [7] considered the Dirichlet series with the complex exponents in the context of a differential equation of infinite order under the constraint that the series $\sum\limits_{k=1}^{+\infty} 1/|\lambda_k|$ should be convergent, but at the same time does not impose any restrictions on the arguments λ_n, which other authors in the topic do. J. Micusiński [9] considered the Dirichlet series of the form (1) by the condition $\lim\limits_{n\to+\infty} \frac{\Re\lambda_n}{\ln n} = +\infty$. E. Hille [8] and some others authors (see, for example [12]) considered series under the condition $\tau(\Lambda) := \varlimsup\limits_{n\to+\infty} \frac{\ln n}{|\lambda_n|} < +\infty$ and, in particular, in the case of $\tau(\Lambda) = 0$ (see also [12], where considered the series of form (1)). Note that the condition $\lim\limits_{n\to+\infty} \frac{\Re\lambda_n}{\ln n} = +\infty$ also implies $\tau(\Lambda) = 0$. The condition $\sum\limits_{k=1}^{+\infty} 1/|\lambda_k|$ implies that $\lim\limits_{n\to+\infty} \frac{n}{|\lambda_n|} = 0$, that is again $\tau(\Lambda) = 0$. G. Peyser [13] considered multiple Dirichlet series with the complex exponents.

For a formal Taylor-Dirichlet series $F \in \mathscr{D}(\Lambda)$ of form (1) we denote $D_\mu(F)$, $D_c(F)$, $D_a(F)$ the set of the existence of maximal term $\mu(z,F) = \max\{|a_k||z|^{m_k} e^{\Re(z\lambda_k)} \colon k \ge 0\}$, the set of the convergence and the set of the absolute convergence of the series (1), respectively; $G_\mu(F) = D_\mu(F) \setminus \partial D_\mu(F)$ and, $G_c(F) = D_c(F) \setminus \partial D_c(F)$ and $G_a(F) = D_a(F) \setminus \partial D_a(F)$ are the domains of the convergence and the absolute convergence, respectively.

It easy to see that $D_a(F) \subset D_c(F) \subset D_\mu(F), \quad G_a(F) \subset G_c(F) \subset G_\mu(F)$.

In article [12], it was investigated the domain of absolute convergence of the series of form

$$F(z) = \sum_{k=1}^{+\infty} \sum_{n=0}^{m_k-1} d_{k,n} k z^n e^{z\lambda_k}, \tag{3}$$

where (λ_k) is a sequence of complex numbers ordered by nondecreasing moduli, $\lambda_k \to \infty$ when $k \to +\infty$, and m_k are natural numbers. Let (ξ_k) be the sequence such that $\xi_{j_k+1} = \xi_{j_k+2} = \ldots = \xi_{j_k+m_k} = \lambda_k$ for all $(k \ge 1)$, and $n_\Lambda(t) = \sum_{|\lambda_k| \le t} 1$, $n_\xi(t) = \sum_{|\xi_k| \le t} 1$ be the counting fuctions of sequences $(|\lambda_k|)$, $(|\xi_k|)$, respectively. It easy to see that $n_\Lambda(t) \le n_\xi(t) \le \sum_{|\lambda_k| \le t} m_k$ for all $t > 0$.

Remark 1.1. 1) If $m_k \le |\lambda_k|$ $(k \ge 1)$ then

$$n_\Lambda(t) \le n_\xi(t) \le \sum_{|\lambda_k|\le t} m_k \le t\cdot n_\Lambda(t) \quad (t>0).$$

2) If $m_k \le n_\Lambda(|\lambda_k|)$ $(k\ge 1)$ then

$$n_\Lambda(t) \le n_\xi(t) \le \sum_{|\lambda_k|\le t} m_k \le \big(n_\Lambda(t)\big)^2 \quad (t>0).$$

3) If $m_k \le e^{|\lambda_k|^\eta}$ $(k\ge 1)$, $\eta\in(0,1)$, then

$$n_\Lambda(t) \le n_\xi(t) \le \sum_{|\lambda_k|\le t} m_k \le e^{t^\eta} n_\Lambda(t) \quad (t>0).$$

All main statements in article [12] are proved under the following two conditions

$$m(\Lambda) := \lim_{k\to+\infty} \frac{m_k}{\lambda_k} = 0, \quad \tau(\xi) := \lim_{t\to+\infty} \frac{\ln n_\xi(t)}{t} = 0.$$

In fact, if $m(\Lambda)=0$, then by Remark 1, 1) we obtain $\tau(\xi)=0 \iff \tau(\Lambda)=0$. By the way, in the other two cases from Remark 1, this statement is also correct.

2. Main results

Let us first consider the case when condition

$$\lambda_k = o(m_k) \quad (k\to+\infty) \tag{4}$$

is fulfilled, that is $m(\Lambda)=\infty$.

Proposition 2.1. *Let* $F\in\mathcal{T}\mathcal{D}(\Lambda,\mathbf{m})$ *be of form* (1). *If condition* (4) *is fulfilled and* $\ln k = o(m_k)$ $(k\to+\infty)$ *then* $G_a(F)=G_c(F)=G_\mu(F)=\mathbb{D}_R := \{z\in\mathbb{C}\colon |z|<R\}$*, where* $\ln R = \varliminf\limits_{k\to+\infty} \frac{-\ln|a_k|}{m_k}$.

Proof. Since, by condition (4)

$$\varlimsup_{k\to+\infty} \frac{\big(\ln|a_k| + m_k \ln|z| + \Re(z\lambda_k)\big)}{m_k} = -\ln R + \ln|z| =: \ln q(z) < 0$$

for all $z\in\mathbb{D}_R$, then for fixed $z\in\mathbb{D}_R$ and given enough small $\varepsilon>0$ such that $\varepsilon_1 = \ln\big(1+\frac{\varepsilon}{q(z)}\big) > 0$ we get $\ln q(z)+\varepsilon_1 < 0 \implies q(z)+\varepsilon < 1$ and

$$|a_k||z|^{m_k} e^{\Re(z\lambda_k)} < (q(z)+\varepsilon)^{m_k} \quad (k\ge k_0).$$

But the condition $\ln k = o(m_k)$ $(k\to+\infty)$ implies that $\sum\limits_{k=k_0}^{+\infty} (q(z)+\varepsilon)^{m_k} < +\infty$. Hence, $z\in G_a(F)$, that is $\mathbb{D}_R \subset G_a(F)$.

If $|z| > R$ then $-\ln R + \ln|z| = \ln q(z) > 0$. For fixed $z \in \mathbb{D}_R$ and for given $\varepsilon \in (0, 1-q(z))$ we put $\varepsilon_1 = -\ln\left(1 - \frac{\varepsilon}{q(z)}\right)$ there exists a sequence $k_j \to +\infty$ $(j \to +\infty)$ such that

$$\frac{\left(\ln|a_k| + m_k \ln|z| + \Re(z\lambda_k)\right)}{m_k} > \ln q(z) - \varepsilon_1 > 0 \quad (k = k_j,\ j \geq 1).$$

But, $q(z) - \varepsilon = q(z)e^{-\varepsilon_1} > 1$. Therefore,

$$|a_k||z|^{m_k} e^{\Re(z\lambda_k)} > (q(z) - \varepsilon)^{m_k} \geq 1 \quad (k = k_j, j \geq 1).$$

So, $|a_k||z|^{m_k} e^{\Re(z\lambda_k)} \not\to 0$ $(k \to +\infty)$, that is $z \notin G_\mu(F)$. Finally, we obtain

$$\mathbb{D}_R \subset G_a(F) \subset G_c(F) \subset G_\mu(F) \subset \mathbb{D}_R,$$

i.e. $G_a(F) = G_c(F) = G_\mu(F) = \mathbb{D}_R$. □

Proposition 2.2. *Let $F \in \mathscr{T}\mathscr{D}(\Lambda, \mathbf{m})$ be of form* (1) *and $F_1 \in \mathscr{D}(\Lambda)$ be of form* (2). *If* $\max\{m_k, \ln k\} = o(\ln|a_k|)$ $(k \to +\infty)$ *then*

$$G_a(F) \subset G_c(F) \subset G_\mu(F) = G_\mu(F_1) = G_a(F_1) = G_c(F_1).$$

Proof of Proposition 2.2. Recall that by definition $F \in \mathscr{T}\mathscr{D}(\Lambda, \mathbf{m}) \implies a_k \to 0$ $(k \to +\infty)$. Thus, for a given $z \in \mathbb{C}$, $\ln|a_k| + m_k \ln|z| \leq (1-\varepsilon)\ln|a_k|$ for arbitrary $\varepsilon \in (0,1)$ and for all $k \geq k_0(z)$. Hence, if $z \in G_\mu(F)$ then there exists $r > 0$ such that $\mathbb{D}_r(z) := \{\tau\colon |\tau - z| \leq r\} \subset G_\mu(F)$. Consider $\varepsilon = \varepsilon(z) := \frac{r}{|z|+r} \in (0, 1/2)$, so $|z - \frac{z}{1-\varepsilon}| = \frac{|z|\varepsilon}{1-\varepsilon} = r$, i.e. $\frac{z}{1-\varepsilon} \in \mathbb{D}_r(z) \subset G_\mu(F)$. Thus,

$$\ln|a_k| + m_k \ln|z| + \Re(z\lambda_k) = (1 + o(1))\ln|a_k| + \Re(z\lambda_k) \leq$$
$$\leq (1-\varepsilon)\ln|a_k| + \Re(z\lambda_k) = (1-\varepsilon)\left(\ln|a_k| + \Re\frac{z\lambda_k}{1-\varepsilon}\right) \to -\infty \quad (k \to +\infty),$$

that is $z \in G_\mu(F)$. So, $G_\mu(F) \subset G_\mu(F_1)$.

And vice versa. If $z \in G_\mu(F_1) \setminus \{0\}$ then there exists $r > 0$ such that $\mathbb{D}_r(z) \subset G_\mu(F_1)$. We will choose again $\varepsilon = \varepsilon(z) := \frac{r}{|z|+r} \in (0, 1/2)$. Then, $|z - \frac{z}{1+\varepsilon}| = \frac{|z|\varepsilon}{1+\varepsilon} = r$, i.e. $\frac{z}{1+\varepsilon} \in \mathbb{D}_r(z) \subset G_\mu(F_1)$. Thus,

$$\ln|a_k| + \Re(z\lambda_k) = \ln|a_k| + (1-\varepsilon)m_k \ln|z| + \Re(z\lambda_k) - (1-\varepsilon)m_k \ln|z| \leq$$
$$\leq (1-\varepsilon)\ln|a_k| + (1-\varepsilon)m_k \ln|z| + \Re(z\lambda_k) =$$
$$= (1-\varepsilon)\left(\ln|a_k| + m_k \ln|z| + \Re\frac{z\lambda_k}{1-\varepsilon}\right) \to -\infty \quad (k \to +\infty),$$

that is $z \in G_\mu(F)$. So, $G_\mu(F_1) \setminus \{0\} \subset G_\mu(F) \setminus \{0\}$.
It remains to note that $0 \in G_\mu(F)$ and $0 \in G_\mu(F_1)$.

Finally, by [15, Corollary 7] (see also [16, Remark 6]) $G_\mu(F) = G_\mu(F_1) = G_a(F_1) = G_c(F_1)$. □

Proposition 2.3. *Let Λ be a sequence of the non-negative real numbers, $F \in \mathscr{T}\mathscr{D}(\Lambda, \mathbf{m})$ be of form* (1) *and $F_1 \in \mathscr{D}(\Lambda)$ be of form* (2)*. If* $\max\{m_k, \ln k\} = o(\ln|a_k|)$ $(k \to +\infty)$ *then*

$$G_a(F) = G_c(F) = G_\mu(F) = G_\mu(F_1) = G_a(F_1) = G_c(F_1) = \Pi_{\alpha_0} := \{z\colon\ \Re z < \alpha_0\},$$

where

$$\alpha_0 := \varliminf_{k\to+\infty} \frac{-\ln|a_k|}{\lambda_k}.$$

Proof of Proposition 2.3. By Proposition 2.2

$$G_a(F) \subset G_c(F) \subset G_\mu(F) = G_\mu(F_1) = G_a(F_1) = G_c(F_1).$$

In the case $\lambda_k \geq 0$ $(k \geq 0)$ (see [14, Proposition 3], [15, Corollary 1]) $G_\mu(F_1) = \Pi_{\alpha_0}$, so also $G_\mu(F) = \Pi_{\alpha_0}$.

It remains to prove that $G_\mu(F) \subset G_a(F)$. Indeed, assume that $\alpha_0 \neq \infty$ and consider a point z_0 such that $\Re z_0 = \alpha_0 - 2\varepsilon$ for an arbitrary $\varepsilon > 0$. Since, by condition, $a_k \to 0$ $(k \to +\infty)$, $\alpha_0 \geq 0$. If $\alpha_0 > 0$ then $\lambda_k < \frac{-\ln|a_k|}{\alpha_0-\varepsilon}$ $(k \geq k_0)$ for given arbitrary $\varepsilon \in (0, \alpha_0/2)$. Then,

$$\ln|a_k| + m_k\ln|z_0| + \lambda_k\Re z_0 < \Big(1 - \frac{\alpha_0 - 2\varepsilon}{\alpha_0 - \varepsilon}\Big)\ln|a_k| + m_k\ln|z_0| =$$
$$= (1 + o(1))\frac{\varepsilon}{\alpha_0 - \varepsilon}\ln|a_k| \leq (1-\varepsilon)\frac{\varepsilon}{\alpha_0 - \varepsilon}\ln|a_k|$$

as $k \to +\infty$. But, by condition, $-\ln|a_k| > \frac{1}{\varepsilon_1}\ln k$ $(k \geq k_1)$, where $\varepsilon_1 > 0$ is arbitrary. We put $\varepsilon_1 = \frac{\alpha_0-\varepsilon}{2(1-\varepsilon)\varepsilon}$. Hence,

$$\ln|a_k| + m_k\ln|z_0| + \lambda_k\Re z_0 < -2\ln k \quad (k \geq k_2).$$

Therefore,

$$\sum_{k=k_2}^{+\infty} |a_k||z_0|^{m_k}e^{\lambda_k\Re z_0} < \sum_{k=k_2}^{+\infty}\frac{1}{k^2} < +\infty,$$

that is $z_0 \in G_a(F)$. So, $G_\mu(F) \subset G_a(F)$ and, finally, $G_\mu(F) = G_a(F)$.

If $\alpha_0 = +\infty$, we consider a point $z_0 \in \mathbb{C}$. Then $\lambda_k < \frac{-\ln|a_k|}{E}$ $(k \geq k_0)$ for given arbitrary $E > 0$ and

$$\ln|a_k| + m_k\ln|z_0| + \lambda_k\Re z_0 < \Big(1 - \frac{1}{E}\Big)\ln|a_k| + m_k\ln|z_0| =$$
$$= (1 + o(1))\frac{E-1}{E}\ln|a_k| \leq -(1-\varepsilon)\frac{E-1}{E}\ln|a_k| \quad (k \to +\infty). \qquad (5)$$

But, by condition, $\ln|a_k| < -\frac{1}{\varepsilon}\ln k$ $(k \geq k_1)$, where ε is arbitrary. Let's choose $E > 0$ and $\varepsilon > 0$ such that $(1-\varepsilon)(E-1) = 2E\varepsilon$. For this, it is enough to take, for example, $\varepsilon = 1/5, E = 2$. Then from inequality (5) we obtain $\ln|a_k| + m_k \ln|z_0| + \lambda_k \Re z_0 < -2\ln k$ $(k \geq k_2)$, hence $\sum\limits_{k=k_2}^{+\infty} |a_k||z_0|^{m_k} e^{\lambda_k \Re z_0} < \sum\limits_{k=k_2}^{+\infty} k^{-2} < +\infty$. This is again $G_\mu(F) \subset G_a(F)$ and $G_\mu(F) = G_a(F) = \mathbb{C}$.

We assume now that $\alpha_0 = 0$. Then, for each point z_0 such that $\Re z_0 = -\varepsilon$ for an arbitrary $\varepsilon > 0$ we get

$$\ln|a_k| + m_k \ln|z_0| + \lambda_k \Re z_0 < \ln|a_k| + m_k \ln|z_0| = (1+o(1))\ln|a_k| \quad (k \to +\infty).$$

Next, it is easy, as above, to obtain that $z_0 \in G_a(F)$. Therefore, $G_\mu(F) \subset G_a(F)$ and $G_\mu(F) = G_a(F) = \Pi_0$. □

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Received by editorial office 8 November 2023

ПРО ОБЛАСТІ ЗБІЖНОСТІ РЯДІВ ТЕЙЛОРА-ДІРІХЛЕ З КОМПЛЕКСНИМИ ПОКАЗНИКАМИ

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У статті розглядаються ряди Діріхле вигляду $F(z) = \sum\limits_{k=0}^{+\infty} a_k e^{z\lambda_k}$, *а також ряди Тейлора-Діріхле вигляду* $F_1(z) = \sum\limits_{k=0}^{+\infty} z^{m_k} a_k e^{z\lambda_k}$, *де* (λ_k) *— деяка послідовність попарно різних комплексних чисел чисел, а* $\big(m_k\big)$ *— послідовність невід'ємних цілих чисел,* $a_k \in \mathbb{C}$. *Основними твердженнями роботи є Твердження 1–3, що стосуються описання областей збіжності, абсолютної збіжності і області існування максимальних членів вказаних вище рядів* F *і* F_1, *а також співвідношень між ними.*

Ключові слова: *ряди Тейлора-Діріхле, абсциса існування максимального члена.*