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ENTIRE FUNCTIONS OF STRONGLY BOUNDED L-INDEX IN DIRECTION

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A concept of entire function of strongly bounded index have been generalized for a multidimensional case. We introduced a class of entire functions of strongly bounded L-index in direction and some properties of this function class are established.

Key words: *entire function, bounded L-index in direction, strongly bounded L-index in direction, directional derivative.*

Shah S. M. and Shah S. N. [1] supposed a generalization of bounded index for entire functions. They introduced entire functions of strongly bounded index and proved that functions of genus zero and having all negative zeros satisfying a one sided growth condition belong to this new class. Using our definition of bounded L-index in direction [2] we extend their definition to a multidimensional case.

Let $L(z),\ z\in\mathbb{C}^n$, be a positive continuous function.

Definition 1 *(see [2]). An entire function of $F(z),\ z\in\mathbb{C}^n$, is called function of bounded L-index in the direction of $\mathbf{b}\in\mathbb{C}^n$, if there exists $m_0\in\mathbb{Z}_+$ such that for $m\in\mathbb{Z}_+$ and every $z\in\mathbb{C}^n$ next inequality is true:*

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial\mathbf{b}^m}\right|\le\max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\right|:0\le k\le m_0\right\},\tag{1}$$

where $\frac{\partial^0 F(z)}{\partial\mathbf{b}^0}=F(z),\ \frac{\partial F(z)}{\partial\mathbf{b}}=\sum_{j=1}^{n}\frac{\partial F(z)}{\partial z_j}b_j,\ \frac{\partial^k F(z)}{\partial\mathbf{b}^k}=\frac{\partial}{\partial\mathbf{b}}\left(\frac{\partial^{k-1}F(z)}{\partial\mathbf{b}^{k-1}}\right),\ k\ge 2.$

The least such integer m_0 is called the L-index in direction $\mathbf{b}$ of function $F(z)$ and is denoted by $N_{\mathbf{b}}(F,L)$. If such m_0 does not exist then we put $N_{\mathbf{b}}(F,L)=\infty$ and F is said of unbounded L-index in direction. If $n=1$ and $\mathbf{b}=1$ then we obtain a definition of entire function of bounded l-index [4] and if $L(z)\equiv 1,\ n=1$ and $\mathbf{b}=1$ then we obtain a definition of entire function of bounded index [3].

Shah S.M. and Shah S.N. [1] shown that there exist functions of bounded index, and of given order ρ and lower order λ provided

$0 \le \lambda \le \rho \le 1$. Their attempts to construct such functions have led us to the remark that a very simple subclass functions of strongly bounded index, of the class functions of bounded index, displays a particularly useful property. If $f(t)$, $t \in \mathbb{C}$, is a function of strongly bounded index and $P(t)$ is a polynomial then $P(t)f(t)$ is a function of strongly bounded index too.

We introduced a concept entire function of bounded L-index in direction [2]. Thus, it is naturally to consider a generalization of strongly bounded index in $\mathbb{C}^n$.

Definition 2 *An entire function of $F(z)$, $z \in \mathbb{C}^n$, is called function of strongly bounded L-index in the direction of $\mathbf{b} \in \mathbb{C}^n$, if there exist quantities $m_0 \in \mathbb{Z}_+$, $r_0 > 0$, $\chi \in (0,1)$ such that for $m \ge m_0 + 1$ and every $z \in \mathbb{C}^n$, $|z| \ge r_0$, next inequality is true*

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le \chi \max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| : 0 \le k \le m_0\right\}. \qquad (2)$$

If $L(z) \equiv 1$ then $F(z)$ is called a function of strongly bounded index in the direction $\mathbf{b}$. The least such integer m_0 is called the strongly L-index in direction $\mathbf{b}$ of function $F(z)$.

For instance, $F(z) = e^{\sum_{j=1}^{n} z_j}$ is a function of strongly bounded index in any direction $\mathbf{b}$. Here $\chi = \frac{1}{2}$, $r_0 = 0$, $m_0 = 1$. But $F(z) = e^{z_n^2 + \sum_{j=1}^{n-1} z_j}$ is a function of strongly bounded index in any direction $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1}, 0)$ and a function of strongly bounded L-index in direction $\mathbf{b} = (\underbrace{0,0,\ldots,0}_{n-1}, b_n)$ where $b_n \ne 0$, $L(z) = |z_n| + 1$. The numbers $\chi = \frac{1}{2}$, $r_0 = 0$, $m_0 = 1$ are same as in previous example. We now state

Theorem 1 *Let $F(z)$, $z \in \mathbb{C}^n$, be entire function of strongly bounded L-index in direction* $\mathbf{b}$. *Then $F(z)$ is of bounded L-index in direction* $\mathbf{b}$,

Proof of Theorem 1. By Definition 2, there exist fixed quantities χ, $0 < \chi < 1$, r_0 and $m_0 \ge 0$ such that (2) holds for all $m \ge> m_0 + 1$ and all $z \in \mathbb{C}^n$ with $|z| > r_0$.

We examine $F(z)$ and its successive directional derivatives in the closed ball

$$|z| \le r_0. \qquad (3)$$

Let $\mathsf{B}=\{z\in\mathsf{C}^n:|z|<r_0\}$ and $\overline{\mathsf{B}}$ is a closure of B and $z^0\in\overline{\mathsf{B}}$ is fixed point. We denote $g_{z^0}(t)\equiv F(z^0+t\mathbf{b})$, $t\in\mathsf{C}$, $D_{z^0}=\overline{\mathsf{B}}\cap\{z^0+t\mathbf{b}:t\in\mathsf{C}\}$, $d=\operatorname{diam}\overline{\mathsf{B}}$. Then on the bounded set D_{z^0} entire function $g_{z^0}(t)$ has finite number of zeros or identical equal to zero. In the last case in view of theorem uniqueness this function is even a zero in general for all $t\in\mathsf{C}$. It is clear that it will be implemented inequality (2) on a set D_{z^0}.

Regarding finite number of zeros for simplification of proof to will consider a case, when this function has only one zero on D_{z^0}. It is easily possible it will be to see from the next proof how the similar reasonings are conducted for the finite number of zeros.

Consequently, let $g_{z^0}(t)$ has one zero in a point a_{z^0} on D_{z^0}. Then there is a derivative $g_{z^0}^{(q_{z^0})}$ such that $g_{z^0}^{(q_{z^0})}(a_{z^0})\neq 0$. So $|g_{z^0}^{(q_{z^0})}(t)|\geq h_{z^0}>0$ in some circle $\overline{K_{z^0}}=\{t\in\mathsf{C}:|t-a_{z^0}|\leq d_{z^0}<\dfrac{d}{\sqrt{n}\,|\mathbf{b}|}\}$. Will notice that these all constants it is possible uniformly to limit, that there exists positive constants q, h such, that $q_{z^0}\leq q$, $h_{z^0}\geq h$, for all $z^0\in\mathsf{C}^n$. Suppose that it is not. Then there is a convergent sequence z_p^0, $p\in\mathsf{N}$ and corresponding it convergent sequences $q_{z_p^0}$ and $a_{z_p^0}$ such, that $z_p^0\to z^*\in\overline{\mathsf{B}}$, $a_{z_p^0}\to a^*\in D_{z^*}$, but $q_{z^0}\to+\infty$. There will be exist an finite point a^*, in which entire function $g_{z^*}(t)$ will have a zero of infinity multiplicity, that it is impossible.

Now we prove that $\sup\{q_{z^0}:z^0\in\overline{\mathsf{B}}\}\equiv q<+\infty$, where q_{z^0} is a multiplicity of all zeros function $g_{z^0}^{\mathbf{b}}=F(z^0+t\mathbf{b})$ in $\overline{K}=\{t\in\mathsf{C}:|t|\leq\dfrac{d}{\sqrt{n}\,|\mathbf{b}|}\}$. On the contrary in view of theorem Montelia we suppose that $q=+\infty$. But $\overline{\mathsf{B}}$ is a compact. Then exists sequence $z_j^0\to z^0\in\overline{\mathsf{B}}$ and sequence of zeros $a_j=a_{z_j}\to a\in\overline{\mathsf{B}}$ of multiplicity $q_j=q_{z_j^0}\to+\infty$ such that corresponding sequence $g_j(t)\equiv g_{z_j}(t)$ uniformly convergences on $overlineK$ to analytic function $g(t)$. Thus point a is a zeros of infinite multiplicity for $g(t)$, that it is not impossible.

Thus we have such estimation $q_{z^0} \le q$, $h_{z^0} \ge h$ for all $z^0 \in \overline{\mathrm{B}}$. Let $L_* = \max\{L(z) : z \in \overline{\mathrm{B}}\}$ and $\mu = \min\{|F(z)| : z \in \overline{\mathrm{B}} \setminus \bigcup_{z^0 \in \mathrm{C}^n} K_{z^0}\}$. Then for all $z \in \overline{\mathrm{B}}$ is true

$$\max\left\{|F(z)|, \frac{1}{q_z! L^{q_z}(z)}\left|\frac{\partial^{q_z} F(z)}{\partial \mathbf{b}^{q_z}}\right|\right\} = \max\left\{|F(z)|, \frac{|g_z^{q_z}(0)|}{q_z! L^{q_z}(z)}\right\} \ge$$

$$\ge \min\left\{\mu, \frac{h_z}{q_z! L_*^{q_z}}\right\} \ge \min\left\{\mu, \frac{h}{q! L_*^{q}}\right\} = T > 0.$$

We choose $\alpha > \frac{1}{|\mathbf{b}|\sqrt{n}}$ and consider a set

$$\overline{G^*} = \overline{\mathrm{B}} \cup \bigcup_{z \in \mathrm{B}} \left\{w \in \mathrm{C}^n : |w - z| \le \frac{\alpha}{L(z)}\right\}.$$

We denote $M = \max\{|F(z)| : z \in \overline{G^*}\}$. According to Cauchy inequality for all $z \in \overline{\mathrm{B}}$ such inequality is true

$$\frac{1}{m! L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| = \frac{|g_z^{(m)}(0)|}{m! L^m(z)} \le \left(\frac{|\mathbf{b}|\sqrt{n} L(z)}{\alpha}\right)^m \frac{1}{L^m(z)} \max\left\{|g_z(\theta)| : |\theta| =\right.$$

$$\left. = \frac{\alpha}{|\mathbf{b}|\sqrt{n} L(z)}\right\} = \left(\frac{|\mathbf{b}\sqrt{n}}{\alpha}\right)^m \max\left\{|F(z + \theta\mathbf{b})| : |z + \theta\mathbf{b} - z| = \frac{\alpha}{L(z)}\right\} \le$$

$$\le \left(\frac{|\mathbf{b}\sqrt{n}}{\alpha}\right)^m \max\left\{|F(w)| : |w - z| = \frac{\alpha}{L(z)}\right\} \le M\left(\frac{|\mathbf{b}\sqrt{n}}{\alpha}\right)^m.$$

Thus for every $z \in \overline{\mathrm{B}}$ we obtain

$$\frac{1}{m! L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le M\left(\frac{|\mathbf{b}|\sqrt{n}}{\alpha}\right)^m = \frac{M(|\mathbf{b}|\sqrt{n})^m}{T\alpha^m} T \le$$

$$\le \frac{M\,|\mathbf{b}\sqrt{n}\,|^m}{T\alpha^m} \max\left\{|F(z)|, \frac{1}{q_z! L^{q_z}(z)}\left|\frac{\partial^{q_z} F(z)}{\partial \mathbf{b}^{q_z}}\right|\right\} \le$$

$$\le \frac{M(|\mathbf{b}\sqrt{n})^m}{T\alpha^m} \max\left\{\frac{1}{p! L^p(z)}\left|\frac{\partial^p F(z)}{\partial \mathbf{b}^p}\right| : 0 \le p \le q\right\}.$$

But $\frac{M}{t(|\mathbf{b}|\sqrt{n}\alpha)^m} \to 0$ at $m \to +\infty$, then we choose $m^* \in \mathrm{N}$ such that $\frac{M}{t(|\mathbf{b}|\sqrt{n}\alpha)^m} \le \chi < 1$ for all $m \ge m^*$. We obtain (2) at $m_0 = n^*$ that it was necessary to prove

$$\frac{|f^{(m)}(z)|}{m!} \leq \chi \cdot \max_{0 \leq j \leq n^*} \left\{ \frac{|f^{(j)}(z)|}{j!} \right\} \quad (m \geq n^* + 1), \tag{4}$$

provided $|z| < r_0$. On the other hand, since $n_0 > s+1$, and $F(z)$ is of strongly bounded L-index in direction **b**, (4) holds for $n > p+1$ and $|z| > r_0$. Hence we can drop the restriction on the size of z and this completes the proof.

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ЦІЛІ ФУНКЦІЇ СИЛЬНО ОБМЕЖЕНОГО L-ІНДЕКСУ ЗА НАПРЯМОМ

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Поняття цілої функції сильно обмеженого індексу узагальнено для багатовимірного випадку. Нами введено клас цілих функцій сильно обмеженого L-індексу за напрямом та встановлено деякі властивості цього класу функцій.

Ключові слова: *ціла функція, обмежений L-індекс за напрямом, сильно обмежений L-індекс за напрямом, похідна за напрямом.*