МАТЕМАТИКА ТА МЕХАНІКА

Математичний аналіз

УДК 517.55
DOI: 10.31471/2304-7399-2021-16(60)-7-10

PROPERTIES OF CLASSES OF SLICE ENTIRE FUNCTIONS AND SLICE HOLOMORPHIC IN THE UNIT BALL FUNCTIONS

A. I. Bandura, L. M. Shegda
Department of Advanced Mathematics; Ivano-Frankivsk National Technical University of Oil and Gas; 15 Karpatska street, Ivano-Frankivsk, 76019, Ukraine; e-mail: andriykopanytsia@gmail.com, shegda.luba@gmail.com

In the paper we investigate properties of class of slice entire functions of several complex variables i.e. these functions are entire on every slice $\{z^0+t\boldsymbol{b}: t\in\mathbb{C}\}$ *for an arbitrary* $z^0\in\mathbb{C}^n$ *and for the fixed direction* $\boldsymbol{b}\in\mathbb{C}^n\backslash\{\mathbf{0}\}$*. For a function* F *from this class we consider a slice function* $g_z(t)=F(z+t\boldsymbol{b})(z\in\mathbb{C}^n, t\in\mathbb{C})$ *and a directional derivative* $\partial_{\boldsymbol{b}}F(z):=g'_z(0), \partial_{\boldsymbol{b}}^pF(z):=\partial_{\boldsymbol{b}}(\partial_{\boldsymbol{b}}^{p-1}F(z)), p\geq 2$*. We show that if a joint continuous function* F *belongs to this class then for any* $p\in\mathbb{N}$ *the function* $\partial_{\boldsymbol{b}}^pF$ *also belongs to the class and is joint continuous. A similar result is also obtained for functions which are slice holomorphic in the unit ball.*

Keywords: *slice entire function, slice holomorphic function, several complex variables, unit ball, directional derivative.*

The presented paper is some addendum to recent papers [1, 5]. There was introduced a notion of boundedness of L-index in direction for two interesting classes of functions: slice entire function of several complex variables and slice holomorphic in the unit ball functions and obtained some criteria of L-index boundedness in direction.

In the following papers [2, 4, 3] Bandura A., Skaskiv O. and Smolovyk L. generalized many known results from theory of bounded index for entire functions. Their new results desribe properties of slice entire and slice holomorphic in the unit ball functions of bounded L-index in direction.

Let us remind some notations from [1, 2, ?]. Let $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}_+^* = [0, +\infty)$, $\mathbf{0} = (0, \dots, 0)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direciton. The slice functions on a line $\{z^0 + t\mathbf{b}: t \in \mathbb{C}\}$ for fixed $z^0 \in \mathbb{C}^n$ we will denote as $g_{z^0}(t) = F(z^0 + t\mathbf{b})$, where $F: \mathbb{C}^n \to \mathbb{C}$.

Let $\widetilde{\mathcal{H}}_{\boldsymbol{b}}^n$ be a class of functions which are holomorphic on every slice $\{z^0 + t\mathbf{b}: t \in \mathbb{C}\}$ for each $z^0 \in \mathbb{C}^n$ and let $\mathcal{H}_{\boldsymbol{b}}^n$ be a class of functions from $\widetilde{\mathcal{H}}_{\boldsymbol{b}}^n$ which are joint continuous. The notation $\partial_{\mathbf{b}} F(z)$ stands for the derivative of the function $g_z(t)$ at the point 0, i.e., for every $p \in \mathbb{N} \partial_{\mathbf{b}}^p F(z) = g_z^{(p)}(0)$, where $g_z(t) = F(z + t\mathbf{b})$ is entire function of complex variable $t \in \mathbb{C}$ for given $z \in \mathbb{C}^n$. In this research, we will often call this derivative as directional derivative because if F is entire function in $\mathbb{C}^n$ then the derivatives of the function $g_z(t)$ matches with directional derivatives of the function F.

The paper [1] contain the following sentences: "Please note that if $F \in \mathcal{H}_{\boldsymbol{b}}^n$ then for every $p \in \mathbb{N}$ $\partial_{\mathbf{b}} F \in \mathcal{H}_{\boldsymbol{b}}^n$. It can be proved by using of Cauchy's formula." This fact was implicitly used in proof of many results from [5, 1, 2, 4, 3]. In the presented paper we give a full proof of this fact.

Theorem 1. *If $F \in \widetilde{\mathcal{H}}_{\boldsymbol{b}}^n$ then for every $p \in \mathbb{N} \partial_{\boldsymbol{b}} F \in \widetilde{\mathcal{H}}_{\boldsymbol{b}}^n$. If $F \in \mathcal{H}_{\boldsymbol{b}}^n$ then for every $p \in \mathbb{N} \partial_{\boldsymbol{b}} F \in \mathcal{H}_{\boldsymbol{b}}^n$.*

Proof. Since F is a slice entire function of several complex variables we can consider a slice function $g_z(t) = F(z + t\mathbf{b})$ as entire function of variable $t \in \mathbb{C}$. For the function g we write integral formula of Cauchy in the following form

$$F(z + t\mathbf{b}) = g_z(t) = \frac{1}{2\pi i} \int_{|\tau|=r} \frac{g_z(\tau)}{\tau} d\tau = \frac{1}{2\pi i} \int_{|\tau|=r} \frac{F(z + \tau\mathbf{b})}{\tau} d\tau.$$

It is valid for any $z \in \mathbb{C}^n, t \in \mathbb{C}, r > 0$. One should observe that $\partial_{\mathbf{b}}^p F(z + t\mathbf{b}) = g_z^{(p)}(t)$ for each $p \in \mathbb{N}$. Indeed,

$$\partial_{\mathbf{b}}^p F(z + t\mathbf{b}) = g_{z+t\mathbf{b}}^{(p)}(0) = \frac{p!}{2\pi i} \int_{|\tau|=r} \frac{g_{z+t\mathbf{b}}(\tau)}{\tau^{p+1}} d\tau = \frac{p!}{2\pi i} \int_{|\tau|=r} \frac{F(z + t\mathbf{b} + \tau\mathbf{b})}{\tau^{p+1}} d\tau. \quad (1)$$

Changing variable $\tau = s - t$, we obtain

$$\partial_{\mathbf{b}}^p F(z + t\mathbf{b}) = \frac{p!}{2\pi i} \int_{|s-t|=r} \frac{F(z + s\mathbf{b})}{(s - t)^{p+1}} ds = \frac{p!}{2\pi i} \int_{|s-t|=r} \frac{g_z(s)}{(s - t)^{p+1}} ds = \\ = g_z^{(p)}(t).$$

Thus, the function $\partial_{\mathbf{b}}^p F$ is also slice entire, i.e. $\partial_{\mathbf{b}} F \in \widetilde{\mathcal{H}}_{\mathbf{b}}^n$.

Let $F \in \mathcal{H}_{\mathbf{b}}^n$. This means that F is continuous in $\mathbb{C}^n$. Then the equation

$$\partial_{\mathbf{b}}^p F(z + t\mathbf{b}) = \frac{p!}{2\pi i} \int_{|\tau|=r} \frac{F(z + t\mathbf{b} + \tau\mathbf{b})}{\tau^{p+1}} d\tau$$

yields that $\partial_{\mathbf{b}}^p F(z + t\mathbf{b})$ is also continuous in $\mathbb{C}^n$ for each $p \in \mathbb{N}$. Thus, the function $\partial_{\mathbf{b}}^p F$ belongs to the class $\mathcal{H}_{\mathbf{b}}^n$.

Theorem 1 is proved.

A similar result is also valid for slice holomorphic function in the unit ball. This class of functions was firstly considered in [5]. A function $F: \mathbb{B}^n \to \mathbb{C}$ is called a slice holomorphic in the unit ball function, if it is analytic in the intersection of every slice $\{z^0 + t\mathbf{b}: t \in \mathbb{C}\}$ with the unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^{\cdot} |z|: = \sqrt{|z|_1^2 + \cdots + |z_n|^2} < 1\}$ for any $z^0 \in \mathbb{B}^n$.

For a given $z \in \mathbb{B}^n$, we denote $S_z = \{t \in \mathbb{C}: z + t\mathbf{b} \in \mathbb{B}^n\}$. Let $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ be a class of functions which are holomorphic on every slices $\{z^0 + t\mathbf{b}: t \in S_{z^0}\}$ for each $z^0 \in \mathbb{B}^n$ and let $\mathcal{H}_{\mathbf{b}}(\mathbb{B}^n)$ be a class of functions from $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$ which are joint continuous.

Theorem 2. *If $F \in \widetilde{\mathcal{H}}_{\boldsymbol{b}}(\mathbb{B}^n)$ then for every $p \in \mathbb{N} \partial_{\boldsymbol{b}} F \in \widetilde{\mathcal{H}}_{\boldsymbol{b}}(\mathbb{B}^n)$. If $F \in \mathcal{H}_{\boldsymbol{b}}(\mathbb{B}^n)$ then for every $p \in \mathbb{N} \partial_{\boldsymbol{b}} F \in \mathcal{H}_{\boldsymbol{b}}(\mathbb{B}^n)$.*

Proof of this theorem is similar to proof of Theorem 1 if we replace $z \in \mathbb{C}^n$ by $z \in \mathbb{B}^n$ and $t \in \mathbb{C}$ by $t \in S_z$. There is only one difference in (1). This equation is valid for all $r \in (0; \frac{1-|z+t\mathbf{b}|}{|\mathbf{b}|})$ if $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{B}^n)$.

Acknowledgment. The first author cordially thanks unknown Referee of paper Bandura A., Skaskiv O.: Slice Holomorphic Functions in Several Variables with Bounded L-Index in Direction, Axioms, **8** (3), Article ID: 88 (2019). doi: 10.3390/axioms8030088for his valuable remark on structure of class of slice entire functions of several complex variables.

This research was funded by the National Research Foundation of Ukraine, 2020.02/0025, 0120U103996.

References

1. Bandura A., Skaskiv O.: Slice Holomorphic Functions in Several Variables with Bounded L-Index in Direction, Axioms, 8 (3), Article ID: 88 (2019). doi: 10.3390/axioms8030088
2. Bandura A.I., Skaskiv O.B.: Some criteria of boundedness of the L-index in direction for slice holomorphic functions of several complex variables, Ukr. Matem. Visn., 16 (2), 154–180 (2019). Engl. transl.: J. Math. Sci., 244 (1), 1-21 (2020). doi: 10.1007/s10958-019-04600-7
3. Andriy Bandura, Oleh Skaskiv, Liana Smolovyk, Slice holomorphic solutions of some directional differential equations with bounded L-index in the same direction. Demonstr. Math. (2019), 52 (1), 482–489. doi: 10.1515/dema-2019-0043.
4. A. I. Bandura A., O. B. Skaskiv, Sum and product of functions having bounded L-index in a direction which are slice holomorphic in the same direction. Prykarpatskyi Visn. Nauk. Tov. Im. Shevchenka. Chyslo. – 2019. – 1(53). – P. 9–20. doi: 10.31471/2304-7399-2019-1(53)-9-20
5. Bandura, A.; Martsinkiv, M.; Skaskiv, O. Slice Holomorphic Functions in the Unit Ball Having a Bounded L-Index in Direction. Axioms 2021, 10 (1), 4. https://doi.org/10.3390/axioms10010004

Стаття надійшла до редакційної колегії 05.09.2021 р.

ВЛАСТИВОСТІ КЛАСІВ ЦІЛИХ НА ЗРІЗКАХ ФУНКЦІЙ ТА ФУНКЦІЙ ГОЛОМОРФНИХ НА ЗРІЗКАХ В ОДИНИЧНІЙ КУЛІ

А. І. Бандура, Л. М. Шегда
Івано-Франківський національний технічний університет нафти і газу;
вул. Карпатська 15, Івано-Франківськ, 76019, Україна;
e-mail: andriykopanytsia@gmail.com, shegda.luba@gmail.com

У цій статті досліджуються властивості класу цілих на зрізках функцій декількох комплексних змінних, тобто функцій, що є цілими на кожній зрізці вигляду $\{z^0+t\mathbf{b}: t\in\mathbb{C}\}$ *для довільного* $z^0\in\mathbb{C}^n$ *і для фіксованого напрямку* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$. *Для функції* F *з такого класу вводяться функція зрізки* $g_z(t)=F(z+t\mathbf{b})\,(z\in\mathbb{C}^n, t\in\mathbb{C})$ *та похідні за напрямком* $\partial_{\mathbf{b}}F(z):=g'_z(0), \partial_{\mathbf{b}}^{p}F(z):=\partial_{\mathbf{b}}(\partial_{\mathbf{b}}^{p-1}F(z)), p\geq 2.$ *Показуємо, що якщо неперервна за сукупністю змінних функція F належить до цього класу, то для будь-якого* $p\in\mathbb{N}$ *функція* $\partial_{\mathbf{b}}^{p}F$ *також належить до того самого класу і також неперервна за сукупністю змінних. Подібний результат також встановлений для функцій, що голоморфні на зрізках в одиничній кулі.*

Ключові слова: *ціла на зрізці функція, голоморфна на зрізці функція, декілька комплексних змінних, одинична куля, похідна за напрямком.*