УДК 621.3:539.6

SOLVING BOUNDARY PROBLEMS OF ELLIPTIC TYPE BY ACCELERATED METHOD OF STATISTICAL TRIALS

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Presented some results of modern computer technology, mathematically based on Laplace and Poisson's Equations. For solving these tasks methods of finite elements, Monte-Carlo, theory of probability, mathematical physics were used.

Key words: *numeral modeling, elliptic equations, random walks, transitional probability, simplex method of rotation.*

"Search for math problems is not our goal"

An effective approach uses a new version of the Monte-Carlo's method in which a posteriori transition probabilities in the scheme of random walks for the first time are replaced by a priori ones. This leads to a significant acceleration of calculations achieved by using a special computational pattern in the form of the simplex element. The successful combination of probabilistic ideas of Monte Carlo's method and bar-centric coordinates of simplex element allows us not to compile and solve large systems of linear algebraic equations. The traditional application of the finite elements' grid in the given area also becomes unnecessary. It is sufficient to provide rotation of the simplex, which translates boundary information to the test point. Both mathematical models and useful computerised applications, which are implementing algorithms of rotation simplex method – RSM, were written and obtained. Computerised technology based on RSM and stationary temperature problem and problems of torsion of elastic rods of complex section was developed and implemented.

The problem of defining a stationary temperature field within a certain area at a given temperature on the boundary of the region is reduced to the solution of the Dirichlet problem for the Laplace equation.

Let us submit a brief summary of the main results of computer analysis of the problem of stationary temperature distribution in a square plate. This problem with geometrically simple region is attractive because one could compare different computational methods and follow the advantages and disadvantages obtained from them.

Suppose you want to find a stationary temperature distribution in a square plate. In both sides of this plate $x = 0$ and $x = 1$ temperature is maintained correspondingly at 0^0 and 100^0 . In the side $y = 0$ temperature increases linearly, while in another side $y = 1$ like a quadratic parabola.

In the same task with 25 nodes, 9 of them inner, was solved by applying finite difference method, followed by solving the resulting system of equations using Cramer's Rule, and iterative method. Solution obtained by Cramer's Rule, was considered as precise. The same result is achieved on 30 iterations of simultaneous displacement method, on iterations of successive displacement method and on 9 iterations of consistent method of upper relaxation. Almost the same precision for arbitrary point of the given region could be obtained by applying the above mentioned RSM method.

We will apply standard program for calculating values of the temperature $T(A)$ in the test points $A_1, A_2, ..., A_9$. Input data are: coordinates of internal nodes $A_1, A_2, ..., A_9$, coordinates of boundary nodes $B_1, B_2, ..., B_{12}$, temperature values $T(B)$ at boundary nodes $B_1, B_2, ..., B_{12}$. Three simplex templates were used for computations: pattern 1 , (B_1, B_4, B_7), pattern 2 (B_1, B_5, B_9) and pattern 3 (B_1, B_6, B_{11}) (Fig. 1).

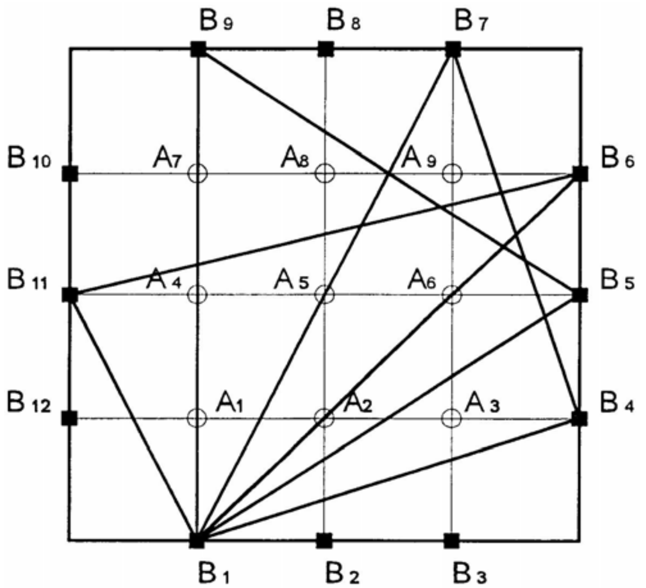

Fig. 1. Points obtained on the plate and computational templates

Computations for each pattern, and their average values are given in Table 1. Relative error δ (%) of the numerical data in the research $0,1 \le \delta \le 2,5$, which indicates a rather high accuracy of the proposed method.

The solution for the problem of torsional stress is Prandtl's function, which satisfies Poisson's equation and is contact in the boundary of the given region. Moreover, in a homogeneous region its value in the boundary equals to zero.

Table 1. *Temperature problem solved by using SRM*

Test Points	Accurate Solution	SRM Pattern 1	SRM Pattern 2	SRM Pattern 3	SRM Arithmetic Mean	Relative Error δ (%)
A_1	23,493	24,375	22,656	23,958	23,663	0,6
A_2	47,879	47,143	49,143	47,591	47,984	0,2
A_3	73,493	74,375	72,656	73,958	73,663	0,2
A_4	21,094	21,964	19,792	23,177	21,644	2,5
A_5	44,531	44,792	44,792	44,705	44,763	0,5
A_6	71,094	71,964	69,792	73,177	71,644	0,7
A_7	16,350	15,208	17,969	15,625	16,267	0,5
A_8	38,058	37,500	40,365	35,482	37,782	0,7
A_9	66,350	65,208	67,969	65,625	66,267	0,1

Let us consider the problem of torsion of a homogeneous prismatic rod. In this article the region was divided into 20 parts. Figure 2 shows calculated points at which values of a stress function were determined. By means of those values both torque moment and maximum tangent torsion were computed. Dirichlet's problem for Poisson's equation was being solved in with the help of grid's method with 31 nodes, 11 of which were internal (Fig. 2).

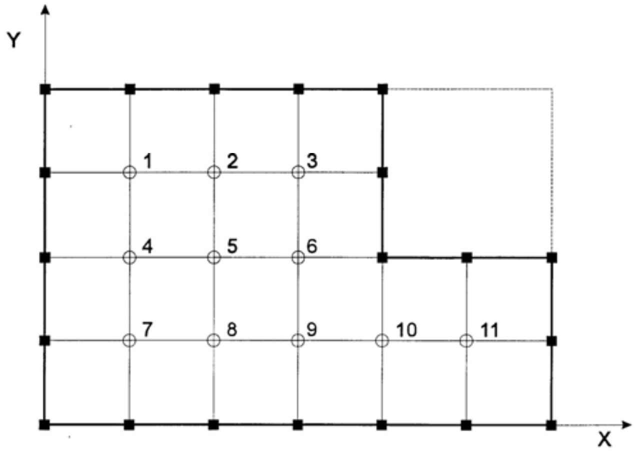

Fig. 2. Shape and cross-section calculated in terms of FDM

The corresponding system of equations was solved by Gauss' method. Since the exact solution for this cross-sectional shape is unknown, the results of calculations were compared with the exact solution for a similar rectangular cross with equal heights. These results were useful for evaluation the order of stress, while both tensions had the same order. When solving the same problem by using method of finite element the research area was divided into 40 rectangular triangles. On forming matrix of stiffness 29 nodes, including 11 interior ones, were taken into the account. For these nodes the augmented matrix of stiffness was of 11x11 order.

Having compared maximum values of tangent stresses calculated by FDM and FEM, we got relative error about 38%, although the values of a function of internal stresses in internal points were equal.

Let us show the way of solving a torsion problem of a rod noncircular cross section (Fig. 3) by using the standard program.

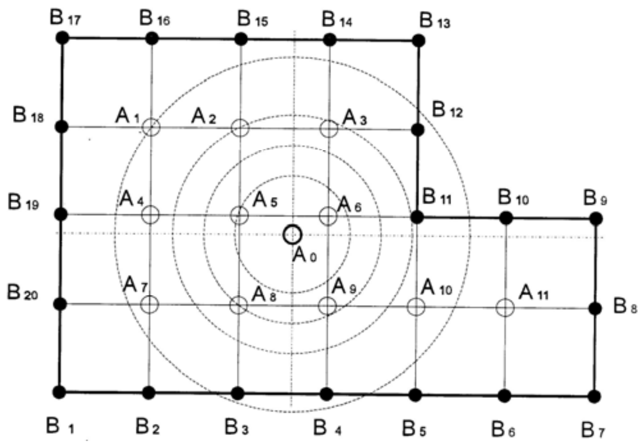

Fig. 3. Shape of the cross section and settlement in terms of SRM

To perform the necessary calculations we must provide the following information:

- the coordinates of research points *A(M)*;
- the coordinates of boundary nodal points *B(N)*;
- the constant value of *H* required for the simplex element.

These calculations are presented in Table 2.

Table 2. Torsion problem solved by using FDM, FEM, SRM

Research Points	FDM, FEM	SRM
A_0	-	2,65909
A_1	1,390	1,75
A_2	1,780	1,87119
A_3	1,410	1,875
A_4	1,780	1,83333
A_5	2,320	2,46968
A_6	1,860	2,55167
A_7	1,410	2,0202
A_8	1,860	2,06817
A_9	1,710	2,33332
A_{10}	1,123	1,5
A_{11}	0,781	1,25

Having compared results obtained by SRM and those by alternative methods, the following conclusions could be made: SRM defines the value of the stress function for any quantity of randomly located dots or in a single point; numerical results are in a strict correspondence of Prandtl's membrane analogy.

Above mentioned methods and results of computer modelling are informative enough so that they can be successfully used in solving problems, which are based on mathematical equations of elliptic type that arise in various applications, such as in solving problems of fluid dynamics, heat conduction, theory of elasticity etc. In particular, two-dimensional Poisson equation is modelling following effects:

– Continuous thermal conductivity

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)=-q\,,$$

– Electrostatic

$$\frac{\partial}{\partial x}\left(\varepsilon\frac{\partial Y}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varepsilon\frac{\partial Y}{\partial y}\right)=-p\,,$$

– Magnetostatics

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu}\cdot\frac{\partial A_z}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\mu}\cdot\frac{\partial A_z}{\partial y}\right)=-j_z\,.$$

Problems of the calculation of cylindrical tubes under the influence of a uniform internal or external pressure are reduced to problems of planar deformation, which are based on differential equation in partial derivatives of the fourth order of the form,

$$\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=0\,,$$

where *φ(x,y)* is the function of stress. In polar coordinates this equation can be represented in the following form.

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2\varphi}{\partial\theta^2}\right)=0\,.$$

It is clear that every solution of the Laplace equation

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2\varphi}{\partial\theta^2}=0$$

is also a solution of the given equation of the fourth order, which makes it possible to apply computer technology for solving some problems, one of them is a problem of stress state of a gas pipeline from corrosion damage.

A new approach to solving boundary value problems of elliptic type offers great opportunities for raising a number of problems in the further theoretical and experimental research and development of modern technologies of computerised implementation.

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Стаття надійшла до редакційної колегії 4.05.2015 р.
Рекомендовано до друку д.т.н., професорм ***Мойсишиним В.М.,***
к.ф.-м.н., доцентом ***Гургулою С.І.***

РОЗВ'ЯЗУВАННЯ КРАЙОВИХ ЗАДАЧ ЕЛІПТИЧНОГО ТИПУ МЕТОДАМИ ПРИСКОРЕНИХ СТАТИСТИЧНИХ ВИПРОБУВАНЬ

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Запропоновано та обґрунтовано комп'ютерну технологію дослідження стаціонарних температурних полів в областях складної форми. Розроблено рекомендації щодо практичного застосування прискорених алгоритмів для дослідження напружень та деформацій кручення стержнів довільного перерізу, а також обговорюється можливість поширення новітніх технологій на широкий клас задач-аналогів.

Ключові слова: *чисельне моделювання, еліптичні рівняння, випадкові блукання, перехідна імовірність, спосіб обертання симплексу.*