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SEMIGROUPS OF LINKED UPFAMILIES

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Given a semigroup S we study right and left zeros, idempotents, the minimal ideal, left cancelable and right cancelable elements of the semigroup $N(S)$ of linked upfamilies and characterize groups G whose extensions $N(G)$ are commutative.

Key words: *semigroup, linked upfamily, idempotent, zero, minimal ideal.*

Introduction

In this paper we investigate the algebraic structure of the extension *N(S)* of a semigroup *S*. The through study of various extensions of semigroups was started in [10] and continued in [1]-[7]. The largest among these extensions is the semigroup $\upsilon(S)$ of all upfamilies on *S*. A family M of non-empty subsets of a set X is called *an upfamily* if for each set $A \in \mathsf{M}$ any subset $B \supset A$ belongs to M. Each family B of non-empty subsets of X generates the upfamily $\langle B \subset X : B \in \mathsf{B} \rangle = \{A \subset X : \exists B \in \mathsf{B}\ (B \subset A)\}$. A family F of non-empty subsets of a set X that is closed under taking supersets and finite intersections is called *a filter*. A filter U is called *an ultrafilter* if $\mathsf{U} = \mathsf{F}$ for any filter F containing U. The family $\beta(X)$ of all ultrafilters on a set X is called *the Stone-Cech compactification of X*, see [11], [12]. An ultrafilter $\langle\{x\}\rangle$, generated by a singleton $\{x\}$, $x \in X$, is called *principal*. We consider $X \subset \beta(X) \subset \upsilon(X)$ if each point $x \in X$ is identified with the principal ultrafilter $\langle\{x\}\rangle$ generated by the singleton $\{x\}$.

It was shown in [10] that any associative binary operation $* : S \times S \to S$ can be extended to an associative binary operation $\circ : \upsilon(S) \times \upsilon(S) \to \upsilon(S)$ by the formula

$$\mathsf{L} \circ \mathsf{M} = \langle \bigcup_{a \in L} a * M_a : L \in \mathsf{L},\ \{M_a\}_{a \in L} \subset \mathsf{M} \rangle$$

for upfamilies $\mathsf{L}, \mathsf{M} \in \upsilon(S)$. In this case the Stone-Cech compactification $\beta(S)$ is a subsemigroup of the semigroup $\upsilon(S)$. The semigroup $\upsilon(S)$ contains many other important extensions of S. In particular, it contains the semigroup *N(S)* of linked upfamilies. A upfamily $\mathsf{L} \in \upsilon(S)$ is called *linked* if intersection $A \cap B$ is non-empty for any sets $A, B \in \mathsf{L}$.

A non-empty subset I of a semigroup S is called *an ideal* (resp. *a right ideal, a left ideal*) if $IS \cup SI \subset I$ (resp. $IS \subset I$, $SI \subset I$). An element z of a semigroup S is called *a zero* (resp. *a left zero*, *a right zero*) in S if $az = za = z$ (resp. $za = z$, $az = z$) for any $a \in S$. An element $a \in S$ is called *an idempotent* if $aa = a$. An ideal $I \subset S$ is called *minimal* if any ideal of S that lies in I coincides with I. By analogy we define minimal left and minimal right ideals of S. The union $K(S)$ of all minimal left (right) ideals of S coincides with the minimal ideal of S, see [11, теор. 2.8]. A semigoup S is said to be *a right zeros semigroup* if $ab = b$ for any $a, b \in S$. A semigroup S is called *right simple* if $aS = S$ for any $a \in S$. An element a of a semigroup S is called *left cancelable (*resp. *right cancelable)* if for any points $x, y \in S$ the equation $ax = ay$ (resp. $xa = ya$) implies $x = y$. This is equivalent to saying that the left (resp. right) shift $l_a : S \to S$, $l_a : x \mapsto ax$, (resp. $r_a : S \to S$, $r_a : x \mapsto xa$) is injective.

1 Zeros and the minimal ideal of the semigroup *N(S)*

For a semigroup S right zeros in *N(S)* admit a simple description. We define a linked upfamily $\mathsf{L} \in N(S)$ to be *shift-invariant* if for every $L \in \mathsf{L}$ and $s \in$ S the sets sL and s^{-1}L={$t \in$ S | st$\in L$} belong to L.

Proposition 1. A linked upfamily $\mathsf{L} \in N(S)$ is a right zero in *N(S)* if and only if L is shift-invariant.

Proof. Assuming that a linked upfamily $\mathsf{L} \in N(S)$ is shift-invariant, we shall show that $\mathsf{M} \circ \mathsf{L} = \mathsf{L}$ for every $\mathsf{M} \in N(S)$. Take any set $F \in \mathsf{M} \circ \mathsf{L}$ and find a set $M \in \mathsf{M}$ and a upfamily $\{L_s\}_{s \in M} \subset \mathsf{L}$ such that $\bigcup_{s \in M} sL_s \subset F$. Since $\mathsf{L} \in N(S)$ is shift-invariant, $\bigcup_{s \in M} sL_s \in \mathsf{L}$ and thus $F \in \mathsf{L}$. This proves the inclusion $\mathsf{M} \circ \mathsf{L} \subset \mathsf{L}$. On the other hand, for every $F \in \mathsf{L}$ and every $s \in$ S we get s$^{-1}F \in \mathsf{L}$ and thus $F \supset \bigcup_{s \in S} s(s^{-1}F) \in \mathsf{M} \circ \mathsf{L}$. This shows that L is a right zero of the semigroup *N(S)*.

Now assume that L is a right zero of *N(S)*. Observe that for every $s \in$ S the equality $< s > \circ \mathsf{L} = \mathsf{L}$ implies $sL \in \mathsf{L}$ for every $L \in \mathsf{L}$.

On the other hand, the equality $\{S\} \circ \mathsf{L} = \mathsf{L}$ implies that for every $L \in \mathsf{L}$ there is a upfamily $\{L_s\}_{s \in S} \subset \mathsf{L}$ such that $\bigcup_{s \in S} sL_s \subset L$. Then for every $s \in$ S the set s^{-1}L={$t \in$ S | st$\in L$} $\supset L_s \in \mathsf{L}$ belong to L witnessing that L is shift-invariant.

By $\overleftrightarrow{\mathrm{N}}(S)$ we denote the set of shift-invariant linked upfamilies in *N(S)*. Proposition 1 implies that $\mathsf{M} \circ \mathsf{L} = \mathsf{L}$ for every $\mathsf{M}, \mathsf{L} \in \overleftrightarrow{\mathrm{N}}(\mathrm{S})$. This means that if $\overleftrightarrow{\mathrm{N}}(S)$ is not empty, then it is a semigroup of right zeros.

Proposition 2. If a semigroup S contains a right zero, then the minimal ideal K(S) of S coincides with the set of all right zeros of S.

Proof. Let Z be the semigroup of all right zeros of S. Then for every $s,t \in S$ and every $z \in Z$ we get t(zs)=(tz)s=zs. Therefore $zs \in Z$ that is $ZS \subset Z$ and Z is a right ideal. It follows from definition of right zeros that *SZ=Z*. This shows that *Z* is an ideal of *S*. It suffices to chek that *Z* lies in each ideal *I* of *S*. Indeed, $Z = IZ \subset IS \subset I$.

Now we find conditions on the semigroup S guaranteeing that the set $\overleftrightarrow{\mathrm{N}}(S)$ is not empty.

Proposition 3. A semigroup S is right simple if and only if {S} is a right zero of *N(S)*.

Proof. Assuming that {S} is a right zero of *N(S)* observe that for every $a \in S$ the equation $\langle\{a\}\rangle \circ \{S\} = \{S\}$ implies that *aS=S*.

On the other hand, if *aS=S* for every $a \in S$, then $\mathsf{M} \circ \{S\}=\{S\}$ for all $\mathsf{M} \in$ *N(S)*. This means that {S} is a right zero of *N(S)*.

Since each group *G* is a right simple semigroup, then *G* contains a right zero by Proposition 3. Therefore Propositions 1 and 2 imply that the minimal ideal *K(N(G))* of semigroup *N(G)* coincides with the set $\overleftrightarrow{\mathrm{N}}(G)$ of all shift-invariant upfamilies of *N(G)*.

A subset *A* of a group *G* is called *self-linked* if $A \cap xA$ is non-empty for each $x \in G$. For a set *A* of a group *G* the upfamily $\{xA \mid x \in G\}$ is orbit of a set *A* under natural left action of a group *G* on the set of subsets of *G*. Proposition 1 implies that each right zero of the semigroup *N(G)* is the union of orbits of self-linked sets of the group *G*.

Proposition 4. The cardinality of the minimal ideal *K(N(G))* of the semigroup *N(G)* over a group *G* of cardinality |G|<8 can be founded from the following table:

G	C_1	C_2	C_3	C_4	$C_2 \oplus C_2$	C_5	C_6	D_3	C_7
K(N(G))	1	1	2	2	2	5	11	17	45

Proof. a) If a group G has cardinality 1 or 2, then *G* is the unique self-lnked subset of *G*. Therefore *K(N(G))*={{*G*}}.

b) In the case $|G| \in \{3,4\}$ a group *G* contains two different orbits of self-linked sets which generated by the sets *G* and *G*\{*e*}, where *e* is the neutral element of *G*. Thus *N(G)* contains two right zeros: {*G*} and $\{G, G \setminus \{g\} \mid g \in G\}$.

c) If |*G*|=5, then *G* is a cyclic group. In this case G contains $C_5^3 = 10$ 3-element sets that generate two different orbits of self-linked sets. Since intersaction of any two 3-element sets is non-empty, then these two orbits (and its union) generate 3 right zeros. Also $N(G)$ contains 2 right zeros $\{G\}$ and $\{G, G\setminus\{g\} \mid g \in G\}$. Therefore $N(G)$ contains 5 right zeros.

d) Let $|G|=6$ and G is isomorphic to a cyclic group $C_6=\{e, a, a^2, a^3, a^4, a^5 \mid a^6=e\}$. In this case G contains two orbits of 3-elements self-linked sets generated by the sets $A=\{e, a, a^3\}$ and $B=\{e, a^2, a^3\}$. Since $A \cap a^2B=\varnothing$, then these two orbits generate two right zeros $\langle gA \mid g \in G\rangle$ and $\langle gB \mid g \in G\rangle$ that contain all sets F of cardinality $|F|>3$. The group C_6 contains three orbits of 4-element subsets generated by the sets $\{e, a, a^2, a^3\}$, $\{e, a, a^3, a^4\}$ and $\{e, a, a^2, a^4\}$. These orbits generate 2^3-1=7 different right zeros. Also $N(C_6)$ contains 2 right zeros $\{C_6\}$ and $\{C_6, C_6\setminus\{g\} \mid g \in C_6\}$. Therefore $|K(N(C_6))|$=2+7+1+1=11.

e) If $|G|=6$ and G is isomorphic to the diedral group $D_3=\{e, a, a^2, b, ab, a^2b \mid a^3=b^2=e, ba= a^2b\}$, then G contains no 3-element self-linked subsets, but all 4-element subsets are self-linked. In this case G contains four orbits of 4-element self-linked sets generated by the sets $\{e, a, a^2, b\}$, $\{e, a, b, ab\}$, $\{e, a^2, b, ab\}$ and $\{e, a^2, ab, a^2b\}$. These orbits generate 2^4-1=15 different right zeros. Also $N(G)$ contains 2 right zeros $\{G\}$ and $\{G, G\setminus\{g\} \mid g \in G\}$. Therefore $|K(N(D_3))|$=15+1+1=17.

f) Let $|G|=7$. Then G is isomorphic to the cyclic group $C_7=\{e, a, a^2, a^3, a^4, a^5, a^6 \mid a^7=e\}$. In this case G contains two orbits of 3-element self-linked sets generated by the sets $A=\{e, a, a^3\}$ and $B=\{e, a^2, a^3\}$. Since $A \cap a^2B=\varnothing$, then these two orbits generate two right zeros $\langle gA \mid g \in G\rangle$ and $\langle gB \mid g \in G\rangle$. The group C_7 has 5 orbits of 4-element self-linked subsets that generate 2^5-1=31 different right zeros. Also C_7 has 3 orbits of 5-element self-linked subsets that generate 2^3-1=7 right zeros. Since the right zero $\langle g\{e,a,a^2,a^3\} \mid g \in C_7\rangle$ does not contain 5-element self-linked set $\{e,a,a^2,a^4,a^5\}$, then the linked upfamily $\langle g\{e,a,a^2,a^3\}, g\{e,a,a^2,a^4,a^5\} \mid g \in C_7\rangle$ also is a right zero of $N(C_7)$. In the same manner $\langle g\{e,a,a^3,a^4\}, g\{e,a^2,a^3,a^4,a^5\} \mid g \in C_7\rangle$ and $\langle g\{e,a^2,a^3,a^5\}, g\{e,a,a^2,a^3,a^5\} \mid g \in C_7\rangle$ are right zeros of $N(C_7)$. Adding right zeros $\{C_7\}$ and $\{C_7, C_7\setminus\{g\} \mid g \in C_7\}$ we conclude that $|K(N(C_7))|$=2+31+7+3+1+1=45.

Now we describe groups G that have (left) zeros and characterize groups G whose extensions $N(G)$ are commutative.

Theorem 1. For a group G the following conditions are equivalent:

1) the semigroup $N(G)$ is commutative;
2) the semigroup $N(G)$ has a zero;
3) the semigroup $N(G)$ has a left zero;

4) G is a cyclic group of cardinality 1 or 2.

Proof. $1) \Rightarrow 2)$ It is easy to see that the linked upfamily $\{G\}$ is shift-invariant and is a right zero of *N(G)* according to Proposition 1. Since the semigroup *N(G)* is commutative, then $\{G\}$ is a zero of *N(G)*.
The implication $2) \Rightarrow 3)$ is trivial.

$\neg 4) \Rightarrow \neg 3)$ If |G|>2, then *N*(*G*) contains at least two shift-invariant linked upfamilies $\{G\}$ and $\{G, G\backslash\{g\} \mid g \in G\}$. According to Proposition 1 it has at least two right zeros and therefore *N*(*G*) has no a left zero.

$4) \Rightarrow 1)$ If $|G|=1$, then $|N(G)|=1$ and *N*(*G*) is commutative. In the case $|G|=2$ the group G is cyclic and the semigroup *N*(*G*) has three elements: two principal ultrafilters and shift-invariant linked upfamily $\{G\}$. Since principal ultrafilters commute with $\{G\}$ and $\{G\}$ is a right zero, then $\{G\}$ is the zero of the semigroup *N*(*G*). Therefore *N*(*G*) is isomorphic to the semigroup G^0 and *N*(*G*) is commutative.

2 Idempotents of the semigroup *N(G)*

In this section we describe some upfamilies of idempotents of the semigroup *N(G)* over a group *G*.

Proposition 5. Let G be a group with the neutral element e and $|G| \geq 2$. For any nonempty subset $A \subset G \setminus \{e\}$, such that $|A \cap \{g, g^{-1}\}| \leq 1$ for each $g \in G$, the linked upfamily $\mathsf{I}_A = \langle \{e,g\},\{e,g^{-1}\} \mid g \in A \rangle$ is an idempotent of the semigroup *N*(*G*).

Proof. First we show that $\mathsf{I}_A \subset \mathsf{I}_A \circ \mathsf{I}_A$. If $L \in \mathsf{I}_A$, then $L \supset \{e,g\}$ or $L \supset \{e,g^{-1}\}$ for some $g \in A$. Since $\{e,g\} = e\{e,g\} \cup g\{e,g^{-1}\} \in \mathsf{I}_A \circ \mathsf{I}_A$ and $\{e,g^{-1}\} = e\{e,g^{-1}\} \cup g^{-1}\{e,g\} \in \mathsf{I}_A \circ \mathsf{I}_A$, then $L \in \mathsf{I}_A \circ \mathsf{I}_A$.

On the other hand, if $L \in \mathsf{I}_A \circ \mathsf{I}_A$, then $L \supset \bigcup_{a \in I} aM_a$, where $\{ I, M_a \mid a \in I \} \subset \mathsf{I}_A$. Since $e \in I$, then $L \supset eM_e = M_e \in \mathsf{I}_A$ and $L \in \mathsf{I}_A$. Therefore $\mathsf{I}_A \circ \mathsf{I}_A = \mathsf{I}_A$ and I_A is an idempotent of the semigroup *N*(*G*).

Proposition 6. If g is an element of order 2 of a group G and $|G| \geq 3$, then the linked upfamily $\mathsf{I}_g = \langle \{e,g\}, G \setminus \{e\}, G \setminus \{g\} \rangle$ is an idempotent of the semigroup *N*(*G*).

Proof. First we prove that $\mathsf{I}_g \subset \mathsf{I}_g \circ \mathsf{I}_g$. If $L \in \mathsf{I}_g$, then $L \supset G \setminus \{e\}$ or $L \supset G \setminus \{g\}$ or $L \supset \{e,g\}$. Since $G \setminus \{e\} = e(G \setminus \{e\}) \cup g(G \setminus \{g\}) \in \mathsf{I}_g \circ \mathsf{I}_g$, $G \setminus \{g\} = e(G \setminus \{g\}) \cup g(G \setminus \{e\}) \in \mathsf{I}_g \circ \mathsf{I}_g$ and $\{e,g\} = \{e,g\}\{e,g\} \in \mathsf{I}_g \circ \mathsf{I}_g$, then $L \in \mathsf{I}_g \circ \mathsf{I}_g$.

Let $L\in \mathsf{I}_g\circ \mathsf{I}_g$, then $L\supset \bigcup_{a\in I} aM_a$, where $\{I, M_a \mid a\in I\}\subset \mathsf{I}_g$. If $e\in I$, then $L\supset eM_e = M_e\in \mathsf{I}_g$ and $L\in \mathsf{I}_g$. It remains to concider the case *I=G\{e}*. Then $g\in I$ and consider the following three cases:

1) if $M_g=\{e,g\}$, then $L\supset \bigcup_{a\in I} aM_a\supset gM_g = M_g\in \mathsf{I}_g$ and $L\in \mathsf{I}_g$;

2) if $M_g = G\setminus\{e\}$, then $L\supset \bigcup_{a\in I} aM_a\supset gM_g = G\setminus\{g\}\in \mathsf{I}_g$ and $L\in \mathsf{I}_g$;

3) if $M_g = G\setminus\{g\}$, then $L\supset \bigcup_{a\in I} aM_a\supset gM_g = G\setminus\{e\}\in \mathsf{I}_g$ and $L\in \mathsf{I}_g$.

Therefore $\mathsf{I}_g\circ \mathsf{I}_g\subset \mathsf{I}_g$ and I_g is an idempotent of the semigroup *N*(*G*).

Proposition 7. Let G be a group with the neutral element e and $|G|\geq 3$. For any subset $A\subset G\setminus\{e\}$, such that $|A\cap\{g,g^{-1}\}|\leq 1$ for each $g\in G$ and $A\neq\{a\}$ where $a^2=e$, the linked upfamily $\mathsf{I}_A^e = \langle G\setminus\{e\},\{e,g\},\{e,g^{-1}\}\mid g\in A\rangle$ is an idempotent of the semigroup *N*(*G*).

Proof. First we show that $\mathsf{I}_A^e\subset \mathsf{I}_A^e\circ \mathsf{I}_A^e$. If $L\in \mathsf{I}_A^e$, then $L\supset\{e,g\}$ or $L\supset\{e,g^{-1}\}$ or $L\supset G\setminus\{e\}$ for some $g\in A$. Consider the case $L\supset G\setminus\{e\}$. If each element of the set A is of order 2, then fix any two different elements $g,h\in A$. Since $g\neq h$, then $gh\neq h^2=e$ and $G\setminus\{e\}=e(G\setminus\{e\})\cup g\{e,h\}$ $\in \mathsf{I}_A^e\circ \mathsf{I}_A^e$. If there exists an element $g\in A$, $g^2\neq e$, then $G\setminus\{e\}=e(G\setminus\{e\})\cup g\{e,g\}$ $\in \mathsf{I}_A^e\circ \mathsf{I}_A^e$. Therefore in this case $L\in \mathsf{I}_A^e\circ \mathsf{I}_A^e$. Let $L\supset\{e,g\}$ or $L\supset\{e,g^{-1}\}$. Since $\{e,g\}=e\{e,g\}\cup g\{e,g^{-1}\}\in \mathsf{I}_A^e\circ \mathsf{I}_A^e$ and $\{e,g^{-1}\}=e\{e,g^{-1}\}\cup g^{-1}\{e,g\}\in \mathsf{I}_A^e\circ \mathsf{I}_A^e$, then $L\in \mathsf{I}_A^e\circ \mathsf{I}_A^e$.

To show that $\mathsf{I}_A^e\circ \mathsf{I}_A^e\subset \mathsf{I}_A^e$ fix any set $L\in \mathsf{I}_A^e\circ \mathsf{I}_A^e$. Then $L\supset \bigcup_{a\in I} aM_a$, where $\{I, M_a\mid a\in I\}\subset \mathsf{I}_A^e$. If $e\in I$, then $L\supset eM_e=M_e\in \mathsf{I}_A^e$ and $L\in \mathsf{I}_A^e$. It remains to concider the case *I=G\{e}*. Let $a\in G\setminus\{e\}$. Lose no generality we can assume that $M_a\in\{G\setminus\{e\},\{e,g\},\{e,g^{-1}\}\mid g\in A\}$. Consider the following three cases:

1) If $M_a=G\setminus\{e\}$, then $aM_a=G\setminus\{a\}$. Since I_A^e contains at least two different 2-element sets, then for some $g\in A$ we have $L\supset \bigcup_{a\in G\setminus\{e\}} aM_a\supset G\setminus\{a\}\supset\{e,g\}\in \mathsf{I}_A^e$ and $L\in \mathsf{I}_A^e$;

2) If $a\in A$ and $M_a=\{e,a^{-1}\}$, then $aM_a=\{e,a\}\in \mathsf{I}_A^e$ and $L\in \mathsf{I}_A^e$;

3) If $M_a\neq\{e,a^{-1}\}$ for any $a\in G\setminus\{e\}$, then $a\in aM_a\subset G\setminus\{e\}$. Therefore $L\supset \bigcup_{a\in G\setminus\{e\}} aM_a = G\setminus\{e\}\in \mathsf{I}_A^e$ and $L\in \mathsf{I}_A^e$.

Therefore $\mathsf{I}_A^e\circ \mathsf{I}_A^e = \mathsf{I}_A^e$ and I_A^e is an idempotent of the semigroup *N*(*G*).

Propositions 5-7 imply the following

Corollary 1. For any infinite group G the semigroup $N(G)$ has $2^{|G|}$ idempotents that are not right zeros.

3 Left cancelable and right cancelable elements of the semigroup *N(S)*

In this section we describe left cancelable and right cancelable elements of the semigroup $N(S)$.

Theorem 2. Let G be a group. A linked upfamily $\mathsf{L}\in N(G)$ is left cancelable in the semigroup $N(G)$ if and only if L is a principal ultrafilter.

Proof. Assume that L is left cancelable in $N(G)$. First we show that L contains some singleton. Assuming the converse, take any point $g_0\in G$ and note that $L(G\backslash\{g_0\})=G$ for any $L\in\mathsf{L}$. To see that this equality holds, take any point $a\in G$, choose two distinct points $b,c\in L$ and find solutions $x,y\in G$ of the equations $bx=a$ and $cy=a$. Since G is right cancellative, then $x\neq y$. Consequently, one of the points x or y is distinct from g_0. If $x\neq g_0$, then $a=bx\in L(G\backslash\{g_0\})$. If $y\neq g_0$, then $a=cy\in L(G\backslash\{g_0\})$. Now for the linked upfamily $\{G, G\backslash\{g_0\}\}\neq\{G\}$, we get $\mathsf{L}\circ\{G, G\backslash\{g_0\}\}=\{G\}=\mathsf{L}\circ\{G\}$, which contradicts the choice of L as a left canceleble element of $N(G)$. Thus L contains some singleton $\{c\}$. Since L is a linked upfamily, then $\mathsf{L}=\langle\{c\}\rangle$ is a principal ultrafilter, which proves the "only if" part of the theorem.

To prove the "if" part, take any principal ultrafilter $\langle\{g\}\rangle$ generated by a singleton $\{g\}\subset G$. We claim that two linked upfamilies $\mathsf{M},\mathsf{L}\in N(G)$ are equal provided $\langle\{g\}\rangle\circ\mathsf{L}=\langle\{g\}\rangle\circ\mathsf{M}$. Indeed, given any set $L\in\mathsf{L}$ observe that $gL\in\langle\{g\}\rangle\circ\mathsf{L}=\langle\{g\}\rangle\circ\mathsf{M}$ and hence $gL=gM$ for some $M\in\mathsf{M}$. The left cancelativity of G implies that $L=M\in\mathsf{M}$, which yields $\mathsf{L}\subset\mathsf{M}$. By the same argument we can also check that $\mathsf{M}\subset\mathsf{L}$.

By the same arguments as in "if" part of Theorem 2 one can prove that principal ultrafilters are right cancelable elements in the semigroup $N(G)$.

If G is a group, then the formula

$$\mathsf{L}\circ\mathsf{M}=\langle\bigcup_{a\in L}a*M_a : L\in\mathsf{L},\ \{M_a\}_{a\in L}\subset\mathsf{M}\rangle$$

implies that the product $\mathsf{L}\circ\mathsf{M}$ of any two linked upfamilies L and M is a principal ultrafilter if and only if both L and M are principal ultrafilters. Therefore we deduce the following proposition.

Proposition 8. For a group G the set $N(G)\setminus\{\langle\{g\}\rangle : g\in G\}$ is an ideal in $N(G)$.

Proposition 9. Let G be a finite group. A linked upfamily $\mathsf{L}\in N(G)$ is right cancelable in the semigroup $N(G)$ if and only if L is a principal ultrafilter.

Proof. Assume that some linked upfamily $\mathsf{M} \in N(G) \setminus \{\langle\{g\}\rangle : g \in G\}$ is right cancelable. This means that the right shift $r_{\mathsf{M}} : N(G) \to N(G)$, $r_{\mathsf{M}} : \mathsf{A} \mapsto \mathsf{A} \circ \mathsf{M}$, is injective. According to Proposition 8, the set $N(G) \setminus \{\langle\{g\}\rangle : g \in G\}$ is an ideal in $N(G)$. Consequently, $r_{\mathsf{M}}(N(G)) = N(G) \circ \mathsf{M} \subset N(G) \setminus \{\langle\{g\}\rangle : g \in G\}$. Since $N(G)$ is finite, r_{M} cannot be injective.

Proposition 10. Let S be a semigroup. A linked upfamily $\mathsf{L} \in N(S)$ is right cancelable in $N(S)$ provided for every $s \in S$ there is a set $L_s \in \mathsf{L}$ such that sL_s $\cap$ tL_t is emptyset for any distinct $s,t \in S$.

Proof. Assume that $\mathsf{A} \circ \mathsf{L} = \mathsf{B} \circ \mathsf{L}$ for two linked upfamilies $\mathsf{A},\mathsf{B} \in N(S)$. First we show that $\mathsf{A} \subset \mathsf{B}$. Take any set $A \in \mathsf{A}$ and observe that the set $\bigcup_{a \in A} aL_a$ belongs to $\mathsf{A} \circ \mathsf{L} = \mathsf{B} \circ \mathsf{L}$. Consequently, there is a set $B \in \mathsf{B}$ and a upfamily of sets $\{M_b\}_{b \in B} \subset \mathsf{L}$ such that

$$\bigcup_{b \in B} bM_b \subset \bigcup_{a \in A} aL_a .$$

It follows from $L_b \in \mathsf{L}$ that $M_b \cap L_b$ is not empty for every $b \in B$.

Since the sets aL_a and bL_b are disjoint for different $a,b \in S$, the inclusion

$$\bigcup_{b \in B} b(M_b \cap L_b) \subset \bigcup_{b \in B} bM_b \subset \bigcup_{a \in A} aL_a$$

implies $B \subset A$ and hence $A \in \mathsf{B}$.

By analogy we can prove that $\mathsf{B} \subset \mathsf{A}$.

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НАПІВГРУПИ ЗЧЕПЛЕНИХ СІМЕЙ

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У статті вивчаються праві і ліві нулі, ідемпотенти, мінімальний ідеал, скоротні зліва і скоротні справа елементи напівгрупи $N(S)$ зчеплених сімей, а також характеризуються групи G, розширення $N(G)$ яких є комутативним.

Ключові слова: *напівгрупа, зчеплена сім'я, ідемпотент, нуль, мінімальний ідеал.*