Алгебра і геометрія

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GENERAL CLASSES OF MUTUALLY INVERSE POLYNOMIALS OF PARTITION

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The paper is devoted to the study of general class of mutually inverse polynomials of partitions.

Key words: *polynomials of partitions, paradeterminant, parapermanent, triangular matrix, recurrence relations.*

Introduction

Polynomials of partitions [1] are widely applied in discrete mathematics. They appear in the number theory [2], algebra (symmetric polynomial theory), combinatorics [3] (e.g., when presenting the sum of divisors of a positive integer with the help of unordered partitions of a positive integer), differentiation of composite functions (Faa di Bruno's formula) [4] etc.

With the help of triangular matrix calculus machinery (see [5], [6]), the present article seeks to study one general class of mutually inverse polynomials of partitions. Their relations with some linear recurrence relations and parafunctions of triangular matrices are determined.

1. Preliminaries.

Let us the first consider some subsidiary notions and statements.

Let K be some number field.

Definition 1. [7]. *A triangular table*

$$A=\begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_n \qquad (1)$$

of numbers from a number field K is called a ***triangular matrix, an element*** a_{11} *is an upper element of this triangular matrix, and a number n is its order.*

To every element a_{ij} of the triangular matrix (1), we correspondent $(i-j+1)$ elements $a_{ik}, k \in \{j,...,i\}$, which are called *derived elements* of triangular matrix, generated by *a key element* a_{ij}. A key element of a triangular matrix is concurrently its derived element. The product of all derived elements generated by a key element a_{ij} is denoted by $\{a_{ij}\}$ and is called *a factor product* of this key element, i.e.

$$\{a_{ij}\} = \prod_{k=j}^{i} a_{ik} \ . \qquad (2)$$

Definition 2. [5]. *If A is the triangular matrix (1), then the following is true:*

$$\begin{aligned} d\det(A) &= \sum_{r=1}^{n} \sum_{p_1+...+p_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{p_1+...+p_s p_1+...+p_{s-1}+1}\}, \\ pper(A) &= \sum_{r=1}^{n} \sum_{p_1+...+p_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{p_1+...+p_s p_1+...+p_{s-1}+1}\}, \end{aligned} \qquad (3)$$

where the summation is over the set of natural solution of the equality $p_1+...+p_r=n$

Definition 3. [7]. *To each element* a_{ij} *of the given triangular matrix (1) we correspond a triangular matrix with this element in the bottom left corner, which we call* ***corner*** *of the given triangular matrix and denoted by* $R_{ij}(A)$.

It is obvious that the corner $R_{ij}(A)$ is a triangular matrix of order $(i-j+1)$. The corner $R_{ij}(A)$ includes only those elements a_{rs} of the triangular matrix (1), the indices of which satisfy the relations $j \le s \le r \le i$.

Below we shall consider that

Theorem 1. [5]. *Decomposition of parafunctions by the elements of the last row. The following identities hold:*

$$d\det(A) = \sum_{s=1}^{n} (-1)^{n+s} \{a_{ns}\} \cdot d\det(R_{s-1,1}),$$

$$pper(A) = \sum_{s=1}^{n} \{a_{ns}\} \cdot pper(R_{s-1,1}).$$

Theorem 2. [5]. *The following identities hold*

$$d\det\begin{pmatrix} k_{11}\cdot x_1 & & & \\ k_{21}\cdot\frac{x_2}{x_1} & k_{22}\cdot x_1 & & \\ \vdots & \cdots & \ddots & \\ k_{n1}\cdot\frac{x_n}{x_{n-1}} & k_{n2}\cdot\frac{x_{n-1}}{x_{n-2}} & \cdots & k_{nn}\cdot x_1 \end{pmatrix}_n =$$

$$pper\begin{pmatrix} k_{11}\cdot x_1 & & & \\ k_{21}\cdot\frac{x_2}{x_1} & k_{22}\cdot x_1 & & \\ \vdots & \cdots & \ddots & \\ k_{n1}\cdot\frac{x_n}{x_{n-1}} & k_{n2}\cdot\frac{x_{n-1}}{x_{n-2}} & \cdots & k_{nn}\cdot x_1 \end{pmatrix}_n =$$

where $x_0 = 1, k_{ij}$ *is some rational function of arguments* i, j *, and* $c(n;\lambda_1,...,\lambda_n)$ *is a rational function dependent also on the coefficients* k_{ij} .

Theorem 3. [5]. *The formulae for inversion of parafanction of triangular matrices are true:*

1) $$b_i = \left(\tau_{sr}\,\frac{a_{s-r+1}}{a_{s-r}}\right)_{1\le r\le s\le i},$$

$$a_i = \left(\tau^{-1}_{s,s-r+1}\,\frac{b_{s-r+1}}{b_{s-r}}\right)_{1\le r\le s\le i},\ i = 1,2,...;$$

2) $$b_i = \left[\tau_{sr}\,\frac{a_{s-r+1}}{a_{s-r}}\right]_{1\le r\le s\le i},$$

$$a_i = (-1)^{i-1}\left(\tau^{-1}_{s,s-r+1}\,\frac{b_{s-r+1}}{b_{s-r}}\right)_{1\le r\le s\le i},\ i = 1,2,....$$

Consider that in the above inverse formulae, the equidistant-from-the – ends coefficients of every row of the matrices are mutually inverse.

2. On one class of polynomials of partitions

The following two theorems are true:

Theorem 4. *Let the polynomials* $y_n(x_1, x_2,..., x_n),\ n = 0,1,...$ *be defined by the recurrence equation*

$$y_n = x_1 y_{n-1} - x_2 y_{n-2} + ... + (-1)^{n-2} x_{n-1} y_1 + (-1)^{n-1} a_n x_n y_0\,, \qquad (4)$$

where $y_0 = 1$*, then the following equalities hold*

$$y_n = d\det\begin{pmatrix} a_1 \cdot x_1 & & & \\ a_2 \cdot \dfrac{x_2}{x_1} & x_1 & & \\ \vdots & \cdots & \ddots & \\ a_n \cdot \dfrac{x_n}{x_{n-1}} & \cdots & \dfrac{x_2}{x_1} & x_1 \end{pmatrix}, \tag{5}$$

$$y_n = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n} (-1)^{n-k}\left(\sum_{i=1}^{n}\lambda_i a_i\right)\frac{(k-1)!}{\lambda_1!\lambda_2!\ldots\lambda_n!}x_1^{\lambda_1}x_2^{\lambda_2}\ldots x_n^{\lambda_n}, \tag{6}$$

where $k = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

Proof. The paradeterminant (5) satisfying the recurrence relation (4) follows from its decomposition by the elements of the last row.

Let us prove the equality (6).

Let the theorem be true for $n = 0,1,\ldots,m$. We shall prove it for $n = m + 1$. We shall find the coefficient for some fixed monomial

$$x_1^{\lambda_1^*}, x_2^{\lambda_2^*}, \ldots, x_m^{\lambda_m^*}, x_{m+1}^{\lambda_{m+1}^*} \tag{7}$$

in the polynomial y_{m+1}.

Two cases are possible:

1) $\lambda_{m+1}^* = 0$ In this case, according to the recurrence relation

$$y_{m+1} = x_1 y_{m-1} - x_2 y_{m-1} + \ldots + (-1)^{m-1} x_m y_1 + (-1)^m a_{m+1} x_{m+1} y_0,$$

the desired coefficient of the monomial (7) can be obtained with the help of the sum of the coefficients corresponding to the summands

$$a(i) = (-1)^{i-1} x_i y_{m-i+1}, \quad i = 1,2,\ldots,m$$

of this relation. It is obvious that the partitions

$$\lambda_1^* + 2\lambda_2^* + \ldots + i(\lambda_i^* - 1) + (i+1)\lambda_{i+1}^* + \ldots + (m-i+1)\lambda_{m-i+1}^* = m - i + 1$$

correspond to the summands $a(i)$, and the coefficients

$$(-1)^{i-1}(-1)^{m-i+1-k-1}\left(\lambda_1^* a_1 + \ldots + (\lambda_i^* - 1)a_i + \ldots + \lambda_m^* a_m\right)\frac{(k-2)!}{\lambda_1^*! \cdot \ldots \cdot (\lambda_i^* - 1)! \cdot \ldots \cdot \lambda_m^*!},$$

where $k = \lambda_1^* + \lambda_2^* + \ldots + \lambda_m^*$, correspond to the pertitions. Considering that $\lambda_{m-i+2}^* = \ldots = \lambda_{m+1}^* = 0$, these coefficients can be written as

$$(-1)^{m-A-1}(B - a_i)\frac{(A-2)!}{\lambda_1^*! \cdot \ldots \cdot (\lambda_i^* - 1)! \cdot \ldots \cdot \lambda_m^*!\, \lambda_{m+1}^*!},$$

where

$$A = \sum_{i=1}^{m+1}\lambda_i^*, \qquad B = \sum_{i=1}^{m+1}\lambda_i^* a_i.$$

Thus, the desired coefficient for the monomial

$x_1^{\lambda_1^*} \cdot x_2^{\lambda_2^*} \cdot \ldots \cdot x_m^{\lambda_m^*} \cdot x_{m+1}^{\lambda_{m+1}^*}$

is equal to

$$\sum_{i=1}^{m+1} (-1)^{m+1-A} (B - a_i)\lambda_i \frac{(A-2)!}{\lambda_1^*! \cdot \ldots \cdot \lambda_{m+1}^*!} = (-1)^{m+1-A} B \frac{(A-1)!}{\lambda_1^*! \cdot \ldots \cdot \lambda_{m+1}^*!}.$$

2) $\lambda_{m+1}^* = 1$ it is obvious that in this case

$\lambda_1^* = \lambda_2^* = \ldots = \lambda_m^* = 0$,

and it follows from the recurrence relation that the desired coefficient is equal to

$$(-1)^m a_{m+1}.$$

But this coefficient can be written as

$$(-1)^{m+1-A} B \frac{(A-1)!}{\lambda_1^*! \cdot \ldots \cdot \lambda_{m+1}^*!},$$

since $A = 1$, $B = a_{m+1}$.

The following theorem can be proved in the same way.

Theorem 5. *Let the polynomials* $y_n(x_1, x_2, \ldots, x_n)$, $n = 0,1,\ldots$ *be defined by the recurrence equation*

$$y_n = x_1 y_{n-1} - x_2 y_{n-2} + \ldots + x_{n-1} y_1 + a_n x_n y_0,$$

where $y_0 = 1$*, then the following is true*

$$y_n = pper\begin{pmatrix} a_1 \cdot x_1 & & & \\ a_2 \cdot \frac{x_2}{x_1} & x_1 & & \\ \vdots & \ldots & \ddots & \\ a_n \cdot \frac{x_n}{x_{n-1}} & \ldots & \frac{x_2}{x_1} & x_1 \end{pmatrix},$$

$$y_n = \sum_{\lambda_1 + 2\lambda_2 + \ldots + n\lambda_n} \left(\sum_{i=1}^n \lambda_i a_i \right) \frac{(k-1)!}{\lambda_1! \lambda_2! \cdot \ldots \cdot \lambda_n!} x_1^{\lambda_1} x_2^{\lambda_2} \cdot \ldots \cdot x_n^{\lambda_n},$$

where $k = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

If in the theorem 4 we set $a_1 = a_2 = \ldots = a_n = 1$ *and* $m = n$, then we get the theorem 2.5.3. from [5]. The parapermanents of these matrices appear when homogenous symmetric polynomials are expressed in terms of power sums (see [5], pg.: 174, 338).

If in the theorem 4 we set $a_i = i, i = 1,\ldots,n$ then we get the expression of Waring's formula written as the paradeterminant of the triangular matrix

$$s_n = d\,det\begin{pmatrix} \sigma_1 & & & & \\ 2\frac{\sigma_2}{\sigma_1} & \sigma_1 & & & \\ \vdots & \cdots & \ddots & & \\ \frac{(n-1)\sigma_{n-1}}{\sigma_{n-2}} & \frac{\sigma_{n-2}}{\sigma_{n-1}} & \cdots & \sigma_1 & \\ n\frac{\sigma_n}{\sigma_{n-1}} & \frac{\sigma_{n-1}}{\sigma_{n-2}} & \cdots & \frac{\sigma_2}{\sigma_1} & \sigma_1 \end{pmatrix},$$

where s_n are symmetric power sums, and σ_i, $i = 1,...,n$ are elementary symmetric polynomials.

An interesting case is when $a_i = ri + s$, where r and s are some rational numbers so that $rs \neq 0$. In this case the following is true.

Corollary 1. *The following equalities are defined by the same polynomials*:

$$y_n = x_1 y_{n-1} - x_2 y_{n-2} + ... + (-1)^{n-2} x_{n-1} y_1 + (-1)^{n-1}(rn+s)x_n y_0,$$

$$y_n = d\,det\begin{pmatrix} (r+s)x_1 & & & \\ (2r+s)\frac{x_2}{x_1} & x_1 & & \\ \vdots & \cdots & \ddots & \\ (rn+s)\frac{x_n}{x_{n-1}} & \cdots & \frac{x_2}{x_1} & x_1 \end{pmatrix},$$

$$y_n = \sum_{\lambda_1 + 2\lambda_2 + ... + n\lambda_n} \frac{(-1)^{n-k}(rn+sk)(k-1)!}{\lambda_1!\,\lambda_2! \cdot ... \cdot \lambda_n!} x_1^{\lambda_1} x_2^{\lambda_2} \cdot ... \cdot x_n^{\lambda_n},$$

where $k = \lambda_1 + \lambda_2 + ... + \lambda_n$, $y_0 = 1$.

A similar corollary is true for the theorem 5:

Corollary 1. *The following equalities are defined by the same polynomials*:

$$y_n = x_1 y_{n-1} - x_2 y_{n-2} + ... + x_{n-1} y_1 + (rn+s)x_n y_0,$$

$$y_n = pper\begin{pmatrix} (r+s)x_1 & & & \\ (2r+s)\frac{x_2}{x_1} & x_1 & & \\ \vdots & \cdots & \ddots & \\ (rn+s)\frac{x_n}{x_{n-1}} & \cdots & \frac{x_2}{x_1} & x_1 \end{pmatrix},$$

$$y_n = \sum_{\lambda_1+2\lambda_2+...+n\lambda_n} (rn+sk)\frac{(k-1)!}{\lambda_1!\lambda_2!\cdot...\cdot\lambda_n!} x_1^{\lambda_1} x_2^{\lambda_2}\cdot...\cdot x_n^{\lambda_n},$$

where $k=\lambda_1+\lambda_2+...+\lambda_n$, $y_0=1$.

3. Inverse polynomials of partitions

Theorem 6. *Let the polynomials* $x_n(y_1,y_2,...,y_n)$, $n=0,1,...$ *be defined by the recurrence relation*

$$x_n = a_n y_1 x_{n-1} - a_n y_2 x_{n-2} + ... + (-1)^{n-2} a_n y_{n-1} x_1 + (-1)^{n-1} a_n y_n x_0,$$

where $x_0=1$*, then the following is true*

$$y_n = d\,det\begin{pmatrix} a_1 y_1 & & & \\ \frac{y_2}{y_1} & a_2 y_1 & & \\ \vdots & \cdots & \ddots & \\ \frac{y_n}{y_{n-1}} & \frac{y_{n-1}}{y_{n-2}} & \cdots & a_n y_1 \end{pmatrix}, \quad (8)$$

$$x_n = \sum_{r=1}^{n} (-1)^{n-r} \sum_{a_1+a_2+...+a_n=n} a_{a_1} a_{a_1+a_2}\cdot...\cdot a_{a_1+a_2+...+a_r} y_{a_1} y_{a_2}\cdot...\cdot y_{a_r},$$

where $a_i>0$, $i=1,2,...,r$.

Proof. The matrix elements of the paradeterminant of the right-hand member of the equality (8) can be written as

$$a_{ij} = \frac{y_{i-j+1}}{y_{i-1}} a_i^{\delta_{ij}},$$

where δ_{ij} is the Kronecker symbol. It follows that according to the definition of the factorial product (2), we have

$$\{a_{ij}\} = \prod_{k=j}^{i} \frac{y_{i-k+1}}{y_{i-k}} a_i^{\delta_{ik}} = y_{i-j+1} a_i.$$

Then we apply the definition of the paradeterminant (3) of the triangular matrix

$$d\,\det(A) = \sum_{r=1}^{n} \sum_{a_1+...+a_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{a_1}+...+a_s, a_{a_1}+...+a_{s-1}+1\}.$$

In the same way, the following theorem for the parapermanents of triangular matrices is proved.

Theorem 7. *Let the polynomials* $x_n(y_1,y_2,...,y_n)$, $n=0,1,...$ *be defined by the recurrence relation*

$$x_n = a_n y_1 x_{n-1} + a_n y_2 x_{n-2} + \ldots + a_n y_{n-1} x_1 + a_n y_n x_0 ,$$

where $x_0 = 1$, *then the following is true*

$$y_n = pper\begin{pmatrix} a_1 y_1 & & & \\ \frac{y_2}{y_1} & a_2 y_1 & & \\ \vdots & \ldots & \ddots & \\ \frac{y_n}{y_{n-1}} & \frac{y_{n-1}}{y_{n-2}} & \cdots & a_n y_1 \end{pmatrix},$$

$$x_n = \sum_{r=1}^{n} \sum_{a_1+a_2+\ldots+a_n=n} a_{a_1} a_{a_1+a_2} \cdot \ldots \cdot a_{a_1+a_2+\ldots+a_r} y_{a_1} y_{a_2} \cdot \ldots \cdot y_{a_r}$$

where $a_i > 0$, $i = 1,2,\ldots,r$.

Corollary 3. *If* $a_i = ri + s$ *where* r *and* s *are some rational numbers so that* $rs \neq 0$ *and* $r \neq s$, *and the equalities hold*

$$x_n = (rn+s)y_1 x_{n-1} - (rn+s)y_2 x_{n-2} + \ldots + (-1)^{n-2}(rn+s)y_{n-1}x_1 + (-1)^{n-1}(rn+s)y_n x_0$$

where $x_0 = 1$, *then the following equalities are true*

$$x_n = d\,det\begin{pmatrix} (r+s)y_1 & & & \\ \frac{y_2}{y_1} & (2r+s)y_1 & & \\ \vdots & \ldots & \ddots & \\ \frac{y_n}{y_{n-1}} & \frac{y_{n-1}}{y_{n-2}} & \cdots & (rn+s)y_1 \end{pmatrix},$$

$$x_n = \sum_{r=1}^{n} (-1)^{n-r} \sum_{a_1+\ldots+a_r=n} \prod_{s=1}^{r} \left(r(a_1 + a_2 + \ldots + a_i) + s \right) y_{a_1} y_{a_2} \cdot \ldots \cdot y_{a_r} ,$$

where $a_i > 0$, $i = 1,2,\ldots,r$.

Remark 1. *If in the theorems 6, 7 instead of the coefficients* $a_i = i, i = 1,\ldots,n$ *we set the inverse* a_i^{-1} *then the polynomials of partitions from the theorems 4, 5 are mutually inverse in accordance with the polynomials of partitions from the theorems 6, 7.*

Remark 2. *If in the theorem 6, 7 we set* $a_i = \frac{1}{i} i, i = 1,2,\ldots,n$ *then we get the equalities with the help of which the elementary symmetric polynomials are defined by the power sums.*

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ЗАГАЛЬНІ КЛАСИ ВЗАЄМНО ОБЕРНЕНИХ МНОГОЧЛЕНІВ РОЗБИТТІВ

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Роботу присвячено вивченню одного загального класу взаємно обернених многочленів розбиттів.

***Ключові слова**: многочлени розбиттів, парадетермінант, параперманент, трикутна матриця, рекурентні співвідношення.*