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ENTIRE FUNCTION OF UNBOUNDED INDEX IN ANY REAL DIRECTION

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We prove that an entire function $F(z_1,z_2)=\cos\sqrt{z_1z_2}$ *is of unbounded index in any complex direction* $\mathbf{b}=(b_1,b_2)\in\mathbb{C}^2\setminus 0$. *But* $F(z_1^0+b_1t,z_2^0+t)$ *is of bounded index for every given* z_1^0, z_2^0 *as function of variable* t. *It is a generalization of our previous result for direction* $(1,1)$.

Keywords: *entire function, bounded* L*-index in direction, directional derivative, unbounded index in any direction.*

1. Introduction.

Let $L(z):\mathbb{C}^n\to\mathbb{R}_+$ be a continuous function, $F:\mathbb{C}^n\to\mathbb{C}$ be an entire function, $g_{z^0}(t):=F(z^0+t\mathbf{b})$, $l_{z^0}(t):=L(z^0+t\mathbf{b})$, $t\in\mathbb{C}$.

Definition *(see [2]). An entire function of* $F(z)$, $z\in\mathbb{C}^n$, *is called a function of bounded* L*-index in a direction of* $\mathbf{b}\in\mathbb{C}^n$, *if there exists* $m_0\in\mathbb{Z}_+$ *such that for all* $m\in\mathbb{Z}_+$ *and every* $z\in\mathbb{C}^n$ *next inequality is true:*

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial\mathbf{b}^m}\right|\le\max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial\mathbf{b}^k}\right|:0\le k\le m_0\right\},$$

where $\dfrac{\partial^0 F(z)}{\partial\mathbf{b}^0}=F(z)$, $\dfrac{\partial F(z)}{\partial\mathbf{b}}=\sum\limits_{j=1}^{n}\dfrac{\partial F(z)}{\partial z_j}b_j$, $\dfrac{\partial^k F(z)}{\partial\mathbf{b}^k}=\dfrac{\partial}{\partial\mathbf{b}}\left(\dfrac{\partial^{k-1}F(z)}{\partial\mathbf{b}^{k-1}}\right), k\ge 2.$

The least such integer m_0 is called a L-index in direction $\mathbf{b}$ of $F(z)$ and is denoted by $N_{\mathbf{b}}(F,L)$. If such m_0 does not exist then we put $N_{\mathbf{b}}(F,L)=\infty$ and F is said of unbounded L-index in direction. We also denote by $N_{\mathbf{b}}(F,L,z^0)$ as L-index in direction $\mathbf{b}$ of function F at a point z^0 that is the least integer m_0 for which inequality (1) is true at $z=z^0$. If $L(z)\equiv 1$ then F is called a function of bounded index in the direction $\mathbf{b}$ and $N_{\mathbf{b}}(F)\equiv N_{\mathbf{b}}(F,1)$ is index in the direction $\mathbf{b}$.

For $\eta>0$, $\mathbf{b}=(b_1,\ldots,b_n)\in\mathbb{C}^n\setminus\{0\}$ and a positive continuous function $L:\mathbb{C}^n\to\mathbb{R}_+$ we define

$$\lambda_1^{\mathbf{b}}(\eta)=\inf\{\inf\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}:t\in\mathbb{C},|t-t_0|\le\frac{\eta}{L(z+t_0\mathbf{b})}\}:t_0\in\mathbb{C},z\in\mathbb{C}^n\},$$

and also

$$\lambda_2^{\mathbf{b}}(\eta)=\sup\{\sup\{\frac{L(z+t\mathbf{b})}{L(z+t_0\mathbf{b})}:t\in\mathbb{C},|t-t_0|\le\frac{\eta}{L(z+t_0\mathbf{b})}\}:t_0\in\mathbb{C},z\in\mathbb{C}^n\}.$$

By $Q_{\mathbf{b}}^n$ we denote the class of functions L which for all $\eta\ge 0$ satisfy the condition

$$0<\lambda_1^{\mathbf{b}}(\eta)\le\lambda_2^{\mathbf{b}}(\eta)<+\infty.$$

We obtained a following assertion.

Theorem 1 *[2] An entire function $F(z)$ is of bounded L-index in direction $\mathbf{b}$ if and only if there exists number $M>0$ such that for all $z^0\in\mathbb{C}^n$ a function $g_{z^0}(t)$ is of bounded l_{z^0}-index with $N(g_{z^0},l_{z^0})\le M<+\infty$ as a function of variable $t\in\mathbb{C}$ and $N_{\mathbf{b}}(F,L)=$ $=\max\{N(g_{z^0},l_{z^0}):z^0\in\mathbb{C}^n\}$.*

In view of Theorem 1, there was a natural *question* [3]: is there an entire function $F(z)$, $z\in\mathbb{C}^n$ and $\mathbf{b}\in\mathbb{C}^n$ such that $N(g_{z^0},l_{z^0})<+\infty$ for all $z^0\in\mathbb{C}^n$, but $N_{\mathbf{b}}(F,L)=+\infty$?

We together with O.B. Skaskiv gave an affirmative answer [3]. There was proved that $F(z_1,z_2)=\cos\sqrt{z_1z_2}$ is of unbounded index in direction $(1,1)$ for $L(z_1,z_2)\equiv 1$. In this paper this result is generalized for any direction.

We need some denotations. If for a given $z^0\in\mathbb{C}^n$ one has $g_{z^0}(t)\ne 0$ for all $t\in\mathbb{C}$, then $G_r^{\mathbf{b}}(F,z^0):=\varnothing$; if for a given $z^0\in\mathbb{C}^n$ we get $g_{z^0}(t)\equiv 0$, then $G_r^{\mathbf{b}}(F,z^0):=\{z^0+t\mathbf{b}:t\in\mathbb{C}\}$. And if for a given $z^0\in\mathbb{C}^n$ we have $g_{z^0}(t)\not\equiv 0$ and a_k^0 are zeros of $g_{z^0}(t)$, then

$$G_r^{\mathbf{b}}(F,z^0):=\bigcup_k\left\{z^0+t\mathbf{b}:|t-a_k^0|\le\frac{r}{L(z^0+a_k^0\mathbf{b})}\right\},r>0.$$

Let

$$G_r^{\mathbf{b}}(F)=\bigcup_{z^0\in\mathbb{C}^n}G_r^{\mathbf{b}}(F,z^0). \tag{2}$$

By $n(r,z^0,t_0,1/F)=\sum_{|a_k^0-t_0|\le r}1$ we denote the counting function of the zero sequence a_k^0.

A following criterion is convenient for a proof whether entire function is of bounded L-index in direction.

Theorem 2 *[2] Let $F(z)$ be an entire in $\mathbb{C}^n$ function, $L \in Q_{\mathbf{b}}^n$. A function $F(z)$ is of bounded L-index in direction $\mathbf{b}$ if and only if:*

1) for every $r>0$ there exists $P=P(r)>0$ such that for each $z \in \mathbb{C}^n \setminus G_r^{\mathbf{b}}(F)$

$$\left| \frac{1}{F(z)} \frac{\partial F(z)}{\partial \mathbf{b}} \right| \le PL(z); \qquad (3)$$

2) for every $r>0$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for every $z^0 \in \mathbb{C}^n$, for which a function $F(z^0+t\mathbf{b}) \not\equiv 0$, and for all $t_0 \in \mathbb{C}$

$$n\left(\frac{r}{|\mathbf{b}|\, L(z^0+t^0\mathbf{b})}, z^0, t_0, \frac{1}{F} \right) \le \widetilde{n}(r). \qquad (4)$$

In a paper of S. Shah and G. Fricke [4] the following proposition is proved.

Lemma 1 *Let $g_0, g_1, \ldots, g_p$ and h be entire functions of bounded index and for every $R \in (0,+\infty)$ there exists number $M = M(R) \in (0,+\infty)$ such that for all $t \in \mathbb{C} \setminus \bigcup_k \{t : |z-c_k| \le R\}$, where c_k are zeros of function g_0, the inequalities hold*

$$|g_j(t)| \le M\, |g_0(t)|, j \in \{1,2,\ldots,p\}. \qquad (5)$$

Then an entire function f, which satisfies an equation

$$g_0(t) f^{(p)}(t) + g_1(t) f^{(p-1)}(t) + \ldots + g_p(t) f(t) = h(t),$$

is of bounded index.

Our main result is next.

Theorem 3 *An entire function*

$$F(z_1, z_2) = \cos\sqrt{z_1 z_2} = \sum_{n=1}^{+\infty} \frac{(-1)^n (z_1 z_2)^n}{(2n)!}$$

is of unbounded index in any direction $\mathbf{b} = (b_1, b_2)$, where $|\mathbf{b}| \ne 0$.

Proof. Let $z^0 = (z_1^0, z_2^0) \in \mathbb{C}^2$, $\mathbf{b} = (b_1, b_2) \in \mathbb{C}^2$ be given, $t \in \mathbb{C}$. We prove that for $z_1 = z_1^0 + b_1 t, z_2 = z_2^0 + b_2 t$ function $F(z_1, z_2)$ is of bounded index as a function of variable t.

Let $F(z_1^0 + b_1 t, z_2^0 + b_2 t) = \cos\sqrt{(z_1^0 + b_1 t)(z_2^0 + b_2 t)} = \cos\sqrt{dt^2 + at + b}$, where $d = b_1 b_2$, $a = z_1^0 b_2 + z_2^0 b_1$, $b = z_1^0 z_2^0$.

For simplicity we denote $f(t) := F(z_1^0 + b_1 t, z_2^0 + b_2 t)$. We evaluate derivatives of this function:

$$f'(t) = -\frac{(2dt+a)\sin\sqrt{dt^2+at+b}}{2\sqrt{dt^2+at+b}},$$

$$f''(t) = -\frac{d\sin\sqrt{dt^2+at+b}}{\sqrt{dt^2+at+b}} - \frac{(2dt+a)^2}{4(dt^2+at+b)}\cos\sqrt{dt^2+at+b} + $$
$$+\frac{(2dt+a)^2\sin\sqrt{dt^2+at+b}}{4(dt^2+at+b)^{3/2}}.$$

Hence, we obtain a differential equation for f:

$$f''(t) + \frac{a^2-4db}{2(2dt+a)(dt^2+at+b)} f'(t) + \frac{(2dt+a)^2}{4(dt^2+at+b)} f(t) = 0. \qquad (6)$$

Rewrite (6) in this look

$$(2dt+a)(dt^2+at+b)f''(t) + \frac{a^2-4db}{2} f'(t) + \frac{(2dt+a)^3}{4} f(t) = 0.$$

Since $g_1(t) = \frac{a^2-4db}{2}$ is constant, then (5) holds for $g_1(t)$. For $g_2(t) = \frac{(2dt+a)^3}{4}$ and $g_0(t) = (2dt+a)(dt^2+at+b)$ we have $\frac{g_2(t)}{g_0(t)} = \frac{(2dt+a)^2}{4(dt^2+at+b)} \to d$ as $t \to +\infty$. Then $\left|\frac{g_2(t)}{g_0(t)}\right| \le M(R)$ for $t \in \mathbb{C} \setminus \bigcup_{k=1}^{3}\{t : |t-c_k| \le R\}$, where c_k are zeros of function $g_0(t) = (2dt+a)(dt^2+at+b)$. Hence, by Lemma 1 the function $f(t)$ is of bounded index.

It remains to prove that function $F(z_1, z_2)$ is of unbounded index in direction **b**. We shall apply Theorem 2, which provides the necessary and sufficient conditions of L-index boundedness in direction. We prove that the condition (4) of this theorem does not hold.

Case 1. Let $b_1 \ne 0$, $b_2 \ne 0$. We denote $a_k = \pi/4 + \pi k$ $(k \in \mathbb{N})$, $\varphi = arg\,(b_1 b_2)$ and put $z^0 = (z_1^0, z_2^0)$, where z_1^0 – arbitrary,

$$z_2^0 = \frac{b_2 z_1^0 + (1-a_k^2)e^{i\varphi/2}}{b_1},\ t_0 = \frac{a_k^2 e^{i\varphi/2} - b_2 z_1^0}{b_1 b_2}.$$

Zeros of function $F(z^0 + t\mathbf{b})$ are found from the equation

$$(z_1^0 + b_1 t)(z_2^0 + b_2 t) = b_1 b_2 t^2 + (z_1^0 b_2 + z_2^0 b_1)t + z_1^0 z_2^0 = (\pi/2 + \pi l)^2, l \in \mathbb{Z}.$$

Consider its roots

$$t_l^{\pm} = \frac{-(b_2 z_1^0 + b_1 z_2^0) \pm \sqrt{(b_2 z_1^0 - b_1 z_2^0)^2 + b_1 b_2 (\pi + 2\pi l)^2}}{2 b_1 b_2}.$$

A condition of belonging zeros $t_l^{\pm}$ to r – neighbourhood at point t_0 has the form

$$r\,|b_1b_2| > \left| a_k^2 e^{i\varphi/2} - b_2 z_1^0 - \frac{-(2b_2z_1^0 + (1-a_k^2)e^{i\varphi/2}) \pm \sqrt{(a_k^2-1)^2 e^{i\varphi} + b_1b_2(\pi+2\pi l)^2}}{2} \right| \Leftrightarrow$$

$$\Leftrightarrow 2r\,|b_1|\cdot|b_2| > |\, a_k^2 + 1 \pm \sqrt{(a_k^2-1)^2 + |b_1b_2|\,(\pi+2\pi l)^2}\,| \Leftrightarrow$$

$$\Leftrightarrow a_k^2 + 1 - 2r\,|b_1|\cdot|b_2| < \sqrt{(a_k^2-1)^2 + |b_1b_2|\,(\pi+2\pi l)^2} < a_k^2 + 1 + 2r\,|b_1|\cdot|b_2|\,.$$

Hence,

$$a_k^4 + 1 + 4r^2 |b_1b_2|^2 + 2a_k^2 - 4r|b_1b_2| - 4r|b_1b_2|\,a_k^2 < a_k^4 - 2a_k^2 + 1 + |b_1b_2|\,(\pi+2\pi l)^2 <$$
$$< a_k^4 + 1 + 4r^2|b_1b_2|^2 + 2a_k^2 + 4r|b_1b_2| + 4r|b_1b_2|\,a_k^2 \Leftrightarrow$$
$$\Leftrightarrow 4r^2|b_1b_2|^2 + 4a_k^2 - 4r|b_1b_2| - 4ra_k^2|b_1b_2| < |b_1b_2|\,(\pi+2\pi l)^2 <$$
$$< 4r^2|b_1b_2|^2 + 4a_k^2 + 4r|b_1b_2| + 4ra_k^2|b_1b_2|\,.$$

Then

$$l \in \left(\frac{2\sqrt{(r^2|b_1b_2|^2 + a_k^2 - r|b_1b_2|(1+a_k^2))/|b_1b_2|} - \pi}{2\pi} ; \frac{2\sqrt{(r^2|b_1b_2|^2 + a_k^2 + r|b_1b_2|(1+a_k^2))/|b_1b_2|} - \pi}{2\pi} \right) \equiv$$
$$\equiv (A_k; B_k)$$

for $r \in \left(0; \dfrac{1}{|b_1b_2|}\right)$. But

$$B_k - A_k = \frac{2r\sqrt{|b_1b_2|}(1+a_k^2)}{\pi\left(\sqrt{r^2|b_1b_2|^2 + a_k^2 - r|b_1b_2|(1+a_k^2)} + \sqrt{r^2|b_1b_2|^2 + a_k^2 + r|b_1b_2|(1+a_k^2)}\right)} \to +\infty$$

as $k \to +\infty$. Then for $r < \dfrac{1}{|b_1b_2|}$ we have $n(r, z^0, t_0, 1/F) \to +\infty$ $(k \to +\infty)$, where z^0, t_0 are defined above.

Case 2. Let $b_1 \neq 0,\ b_2 = 0$.

Now we denote $a_k = \pi/4 + \pi k$ $(k \in \mathrm{N})$, $\varphi = arg\,(b_1)$ and put $z^0 = (z_1^0, z_2^0)$, where $z_1^0 = e^{i\varphi}$, $z_2^0 = a_k^2 \cdot e^{-i\varphi}$, $t_0 = 1$. Zeros of function $F(z^0 + t\mathbf{b})$ are found from the equation

$$(z_1^0 + b_1 t) z_2^0 = z_2^0 b_1 t + z_1^0 z_2^0 = (\pi/2 + \pi l)^2, l \in \mathrm{Z}.$$

Consider its root $$t_l = \frac{(\frac{\pi}{2} + \pi l)^2 - z_1^0 z_2^0}{b_1 z_2^0}.$$

A condition of belonging zero t_l to r – neighbourhood at point t_0 has the form

$$|t_l - t_0| < r \Leftrightarrow$$

$$\Leftrightarrow t_0 - r < \frac{(\frac{\pi}{2}+\pi l)^2 - z_1^0 z_2^0}{b_1 z_2^0} < t_0 + r.$$

We remark that $t_0,\ r,\ b_1 z_2^0,\ z_1^0 z_2^0 \in \mathbb{R}$. Hence,

$$\frac{1}{\pi}\sqrt{(t_0 - r)b_1 z_2^0 + z_1^0 z_2^0} - \frac{1}{2} < l < \frac{1}{\pi}\sqrt{(t_0 + r)b_1 z_2^0 + z_1^0 z_2^0} - \frac{1}{2}.$$

Then

$$l \in \left(\frac{1}{\pi}\sqrt{(t_0 - r)b_1 z_2^0 + z_1^0 z_2^0} - \frac{1}{2}; \frac{1}{\pi}\sqrt{(t_0 + r)b_1 z_2^0 + z_1^0 z_2^0} - \frac{1}{2}\right) \equiv (A_k; B_k)$$

for $r \in (0;1)$. But

$$B_k - A_k = \frac{1}{\pi}\cdot\frac{2rb_1 z_2^0}{\sqrt{(t_0 - r)b_1 z_2^0 + z_1^0 z_2^0} + \sqrt{(t_0 + r)b_1 z_2^0 + z_1^0 z_2^0}} =$$

$$= \frac{1}{\pi}\cdot\frac{2r\,|\,b_1\,|\,a_k^2}{\sqrt{(1-r)\,|\,b_1\,|\,a_k^2 + a_k^2} + \sqrt{(1+r)\,|\,b_1\,|\,a_k^2 + a_k^2}} \to +\infty$$

as $k \to +\infty$. Then for $r < 1$ we have $n(r, z^0, t_0, 1/F) \to +\infty\ \ (k \to +\infty)$, where z^0, t_0 are defined above.

Thus, the function $\cos\sqrt{z_1 z_2}$ is of unbounded index in direction $\mathbf{b}$.

Early we obtained an assertion

Proposition 1 *[1] Let* $\mathbf{b} \in \mathbb{C}^n$ *be a given direction,* $A_0 \subset \mathbb{C}^n$ *such that its closure* $\overline{A}_0 = \{z \in \mathbb{C}^n : \langle z, c\rangle = 1\}$, *where* $\langle c, \mathbf{b}\rangle \neq 0$. *An entire function* $F(z)$ *is bounded L-index in direcrion* $\mathbf{b}$ *if and only if there exists a number* $M > 0$ *such that for all* $z^0 \in A_0$ *a function* $g_{z^0}(t)$ *is bounded* l_{z^0}*-index with* $N(g_{z^0}, l_{z^0}) \le M < +\infty$ *and* $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A_0\}$.

Remark 1 *Theorem 3 implies that a set* A *in Proposition 1 in the general case can not be reduced to a finite set.*

References

1. Bandura A.I. Sufficient sets for boundedness L-index in direction for entire functions / A.I. Bandura, O.B. Skaskiv // Mat. Stud. – 2007. – **30**, no 2. – P. 177-182.
2. Bandura A.I. Entire functions of bounded L-index in direction / A.I. Bandura, O.B. Skaskiv // Mat. Stud. – 2007. – **27**, no. 1. – P. 30-52 (in Ukrainian).
3. Bandura A.I. Entire functions of bounded and unbounded index in direction / A.I. Bandura, O.B. Skaskiv // Mat. Stud. – 2007. – **27**, no. 2. – P. 211-215 (Ukrainian).

4. Fricke G.H. On bounded value distribution and bounded index, Nonlinear Anal / G.H. Fricke, S.M. Shah. – 1978. – **2**, no 4. – P. 423-435.

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ЦІЛА ФУНКЦІЯ НЕОБМЕЖЕНОГО ІНДЕКСУ ЗА БУДЬ-ЯКИМ ДІЙСНИМ НАПРЯМКОМ

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Доведено, що ціла функція $\cos\sqrt{z_1z_2}$ *є необмеженого індексу за будь-яким дійсним напрямком* $\mathbf{b}=(b_1,b_2)\in\mathbb{C}^2\setminus 0$. *Але* $F(z_1^0+b_1t, z_2^0+t)$ *– обмеженого індексу для будь-яких фіксованих* z_1^0, z_2^0 *як функція змінної* t. *Отриманий результат є узагальненням відповідного твердження для напрямку* $(1,1)$.

Ключові слова: *ціла функція, обмежений* L*-індекс за напрямком, похідна за напрямком, необмежений індекс за будь-яким напрямком.*