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SUM AND PRODUCT OF FUNCTIONS HAVING BOUNDED *L*-INDEX IN A DIRECTION WHICH ARE SLICE HOLOMORPHIC IN THE SAME DIRECTION

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In the paper we investigate slice holomorphic functions $F:\mathbb{C}^n\to\mathbb{C}$ *having bounded L-index in a direction, i.e. these functions are entire on every slice* $\{z^0+t\mathbf{b}: t\in\mathbb{C}\}$ *for an arbitrary* $z^0\in\mathbb{C}^n$ *and for the fixed direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$*, and* $(\exists m_0\in\mathbb{Z}_+)$ $(\forall m\in\mathbb{Z}_+)$ $(\forall z\in\mathbb{C}^n)$ *the following inequality holds*

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m!L^m(z)}\le \max_{0\le k\le m_0}\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)},$$

where $\mathbf{L}:\mathbb{C}^n\to\mathbb{R}_+$ *is a positive continuous function,* $\partial_{\mathbf{b}}F(z)=\frac{d}{dt}F(z+t\mathbf{b})|_{t=0}$, $\partial_{\mathbf{b}}^p F=\partial_{\mathbf{b}}(\partial_{\mathbf{b}}^{p-1}F)$ *for* $p\ge 2$. *Our objects of investigations are sum and product of the functions from this class. There are established sufficient conditions providing the boundedness of* L*-index in the same direction for these operations. In particular, the multiplication of functions from this class is closed operation.*

Key words: *bounded index, bounded* L*-index in direction, slice function, holomorphic function, bounded* l*-index, sum.*

1. Introduction. Here we continue our investigations initialized in [1, 2, 3] and we apply our previous results to investigate product and sum of functions from the class introduced here.

Let us remind some notations from [1, 2, 3]. Let $\mathbb{R}_+=(0,+\infty)$, $\mathbb{R}_+^*=[0,+\infty)$, $\mathbf{0}=(0,\ldots,0)$, $\mathbf{b}=(b_1,\ldots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ be a given direciton, $L:\mathbb{C}^n\to\mathbb{R}_+$ be a continuous function, $F:\mathbb{C}^n\to\mathbb{C}$ an entire function. The slice functions on a line $\{z^0+t\mathbf{b}:t\in\mathbb{C}\}$ for fixed $z^0\in\mathbb{C}^n$ we will denote as $g_{z^0}(t)=F(z^0+t\mathbf{b})$ and $l_{z^0}(t)=L(z^0+t\mathbf{b})$. Besides, we denote by $\langle a,c\rangle=\sum_{j=1}^n a_j\overline{c_j}$ the Hermitian scalar product in $\mathbb{C}^n$, where $a,\ c\in\mathbb{C}^n$.

Let $\widetilde{\mathrm{H}}_{\mathbf{b}}^n$ be a class of functions which are holomorphic on every slices $\{z^0+t\mathbf{b}:t\in\mathbb{C}\}$ for each $z^0\in\mathbb{C}^n$ and let $\mathrm{H}_{\mathbf{b}}^n$ be a class of functions from $\widetilde{\mathrm{H}}_{\mathbf{b}}^n$ which are joint continuous. The notation $\partial_{\mathbf{b}}F(z)$ stands for the derivative of the function $g_z(t)$ at the point 0, i.e. for every $p\in\mathbb{N}$ $\partial_{\mathbf{b}}^pF(z)=g_z^{(p)}(0)$, where $g_z(t)=F(z+t\mathbf{b})$ is entire function of complex variable $t\in\mathbb{C}$ for given $z\in\mathbb{C}^n$. It is easy to check that for any $p\in\mathbb{N}$ the derivative $\partial_{\mathbf{b}}^pF$ is also joint continuous.

A function $F\in\mathrm{H}_{\mathbf{b}}^n$ is said [1] to be of *bounded L-index in the direction* $\mathbf{b}$, if there exists $m_0\in\mathbb{Z}_+$ such that for all $m\in\mathbb{Z}_+$ and each $z\in\mathbb{C}^n$ inequality

$$\frac{|\partial_{\mathbf{b}}^mF(z)|}{m!L^m(z)}\le\max_{0\le k\le m_0}\frac{|\partial_{\mathbf{b}}^kF(z)|}{k!L^k(z)},\tag{1}$$

is true. The least integer number m_0, obeying (1), is called the L-index in the direction $\mathbf{b}$ of the function F and is denoted by $N_{\mathbf{b}}(F,L)$. For $n=1$, $\mathbf{b}=1$, $L(z)=l(z)$, $z\in\mathbb{C}$ inequality (1) defines a function of bounded l-index with the l-index $N(F,l)\equiv N_1(F,l)$ [8, 9, 13], and if in addition $l(z)\equiv1$, then we obtain a definition of index boundedness with index $N(F)\equiv N_1(F,1)$ [10, 11]. It is also worth to mention paper [12], where there is introduced the concept of generalized index. It is quite close to the bounded l-index. If our function is entire in the whole n-dimensional complex space then the definition matches with definition of function of bounded L-index in direction, introduced in [4].

Let $N_{\mathbf{b}}(F,L,z^0)$ stands for the L-index in the direction $\mathbf{b}$ of the function F at the point z^0, i.e., it is the least integer m_0, for which inequality (1) is satisfied at this point $z=z^0$. By analogy, the notation $N(f,l,z^0)$ is defined if $n=1$, i.e. in the case of functions of one variable.

Note that the positivity and continuity of the function L are weak restrictions to deduce constructive results. Thus, we assume additional restrictions by the function L.

Let us denote

$$\lambda_{\mathbf{b}}(\eta)=\sup_{z\in\mathbb{C}^n}\sup_{t_1,t_2\in\mathbb{C}}\left\{\frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})}:|t_1-t_2|\le\frac{\eta}{\min\{L(z+t_1\mathbf{b}),L(z+t_2\mathbf{b})\}}\right\}.$$

By $Q_{\mathbf{b}}^n$ we denote a class of positive continuous function $L:\mathbb{C}^n\to\mathbb{R}_+$, satisfying the condition

$$(\forall\eta\ge 0):\lambda_{\mathbf{b}}(\eta)<+\infty, \tag{2}$$

Remark 1 *Note that papers [1, 3] contain results where we use joint continuity. But in this paper our results do not require joint continuity, i.e. they are valid for functions from* $\widetilde{\mathrm{H}}_{\mathbf{b}}^n$ *Therefore, all proofs literally match with the proofs for entire functions of bounded* L*-index in direction.*

We need the following results.

Theorem 1 ([2]). *Let* $L\in Q_{\mathbf{b}}^n$. *A function* $F\in\widetilde{\mathrm{H}}_{\mathbf{b}}^n$ *has bounded* L*-index in the direction* $\mathbf{b}$ *if and only if for any* $r_1>0$, $r_2>0$ $(r_1<r_2)$, *there exists* $P_1=P_1(r_1,r_2)\ge 1$ *such that for every* $z^0\in\mathbb{C}^n$

$$\max\{|F(z^0+t\mathbf{b})|:|t|=\frac{r_2}{L(z^0)}\}\le P_1\max\{|F(z^0+t\mathbf{b})|:|t|=\frac{r_1}{L(z_0)}\}. \tag{3}$$

Denote

$$G_r(F):=G_r^{\mathbf{b}}(F):=\bigcup_{z:F(z)=0}\{z+t\mathbf{b}:|t|<r/L(z)\}.$$

By $n_{z^0}(r,F)=n_{\mathbf{b}}(r,z^0,1/F):=\sum_{|a_k^0|\le r}1$ we denote counting function of zeros a_k^0 for the slice function $F(z^0+t\mathbf{b})$ in the disc $\{t\in\mathbb{C}:|t|\le r\}$. If for given $z^0\in\mathbb{C}^n$ and for all $t\in\mathbb{C}$ $F(z^0+t\mathbf{b})\equiv 0$, then we put $n_{z^0}(r)=-1$. Denote $n(r)=\sup_{z\in\mathbb{C}^n}n_z(r/L(z))$.

Theorem 2 ([2]). *Let* $F\in\widetilde{\mathrm{H}}_{\mathbf{b}}^n$, $L\in Q_{\mathbf{b}}^n$. *If the function* F *has bounded* L*-index in the direction* $\mathbf{b}$, *then*

1) for each $r>0$ *there exists* $P=P(r)>0$ *such that for every* $z\in\mathbb{C}^n\setminus G_r^{\mathbf{b}}(F)$

$$\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right|\le PL(z); \tag{4}$$

2) for any $r>0$ *there exists* $\widetilde{n}(r)\in\mathbb{Z}_+$ *such that for all* $z^0\in\mathbb{C}^n$ *such that* $F(z^0+t\mathbf{b})\not\equiv 0$ *one has*

$$n_{\mathbf{b}}\left(\frac{r}{L(z^0)},z^0,\frac{1}{F}\right)\le\widetilde{n}(r). \tag{5}$$

Theorem 3 ([2]). *Let* $L \in Q_{\mathbf{b}}^{n}$, $F \in \widetilde{\mathrm{H}}_{\mathbf{b}}^{n}$. *If the following conditions are satisfied*

1) there exists $r_1 > 0$ *such that* $n(r_1) \in [-1;\infty)$;

2) there exist $r_2 > 0$, $P > 0$ *such that* $2r_2 \cdot n(r_1) < r_1/\lambda_{\mathbf{b}}(r_1)$ *and for all* $z \in \mathbb{C}^n \setminus G_{r_2}(F)$ *inequality (4) is true;*

then the function F *has bounded* L*-index in the direction* $\mathbf{b}$.

2. Product of functions of bounded L-index in direction. Now we consider an application of Theorem 2. The following proposition can be obtained using similar considerations as in the case of entire functions [7].

Proposition 1. *Let* $L \in Q_{\mathbf{b}}^{n}$, $F \in \widetilde{\mathrm{H}}_{\mathbf{b}}^{n}$ *be a function of bounded* L*-index in the direction* $\mathbf{b}$, $\Phi \in \widetilde{\mathrm{H}}_{\mathbf{b}}^{n}$ *and* $\Psi(z) = F(z)\Phi(z)$. *The function* $\Psi(z)$ *is of bounded* L*-index in the direction* $\mathbf{b}$ *if and only if the function* $\Phi(z)$ *is of bounded* L*-index in the direction* $\mathbf{b}$.

Proof. The similar result was obtained for entire functions of bounded L-index in direction in [7]. Our proof is similar to proof for entire functions in [7] but now we use Theorem 2, deduced for functions holomorphic on the slices. Since an analytic function $F(z)$ has bounded L-index in the direction $\mathbf{b}$, by Theorem 2 for every $r > 0$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for all $z^0 \in \mathbb{C}^n$, satisfying $F(z^0 + t\mathbf{b}) \not\equiv 0$, the estimate $n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{F}\right) \le \widetilde{n}(r)$ holds. Hence,

$$n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{\Phi}\right) \le n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{\Psi}\right) \le n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{\Phi}\right) + \widetilde{n}(r).$$

Thus, condition 2 of Theorem 2 either holds or does not hold for functions $\Psi(z)$ and $\Phi(z)$ simultaneously. If $\Phi(z)$ has bounded L-index in the direction $\mathbf{b}$, then for every $r > 0$ there exist numbers $P_F(r) > 0$ and $P_\Phi(r) > 0$ such that $\left|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}\right| \le P_f(r)L(z)$, $\left|\frac{\partial_{\mathbf{b}} \Phi(z)}{\Phi(z)}\right| \le P_\Phi(r)L(z)$ for each $z \in (\mathbb{C}^n \setminus G_r^{\mathbf{b}}(F)) \cap (\mathbb{C}^n \setminus G_r^{\mathbf{b}}(\Phi))$. Since

$$\mathbb{C}^n \setminus G_r^{\mathbf{b}}(\Psi) \subset (\mathbb{C}^n \setminus G_r^{\mathbf{b}}(F)) \cap (\mathbb{C}^n \setminus G_r^{\mathbf{b}}(\Phi)),$$

$$\left|\frac{\partial_{\mathbf{b}} \Psi(z)}{\Psi(z)}\right| \le \left|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}\right| + \left|\frac{\partial_{\mathbf{b}} \Phi(z)}{\Phi(z)}\right|,$$

for all $z \in \mathbb{C}^n \setminus G_r^{\mathbf{b}}(\Psi)$ we have $\left|\frac{\partial_{\mathbf{b}} \Psi(z)}{\Psi(z)}\right| \le (P_F(r) + P_\Phi(r))L(z)$, i.e. by Theorem 3 the function $\Psi(z)$ is of bounded L-index in the direction $\mathbf{b}$.

On the contrary, let $\Psi(z)$ be of bounded L-index in the direction $\mathbf{b}$, $r > 0$. At first we show that for every $z^0 \in \mathbb{C}^n \setminus G_r^{\mathbf{b}}(F)$ $(r > 0)$ and for every $\widetilde{d}^{\,k} = z^0 + d_k^0\mathbf{b}$, where d_k^0 are zeros of function $\Phi(z^0 + t\mathbf{b})$, we have

$$|z^0-\widetilde{d}^k|>\frac{r|\mathbf{b}|}{2L(z^0)\lambda_{\mathbf{b}}(r)}. \qquad (6)$$

On the other hand, let there exist $z^0\in\mathbb{C}^n\setminus G_r^{\mathbf{b}}(\Phi)$ and $\widetilde{d}^k=z^0+d_k^0\mathbf{b}$ such that $|z^0-\widetilde{d}^k|\le\frac{r|\mathbf{b}|}{2L(z^0)\lambda_{\mathbf{b}}(r)}$. Then by the definition of $\lambda_{\mathbf{b}}$ we have the next estimate $L(\widetilde{d}^k)\le\lambda_{\mathbf{b}}(r)L(z^0)$, and hence $|z^0-\widetilde{d}^k|=|\mathbf{b}|\cdot|d_k^0|\le\frac{r|\mathbf{b}|}{2L(\widetilde{d}^k)}$, i.e. $|d_k^0|\le\frac{r}{2L(\widetilde{d}^k)}$, but it contradicts $z^0\in\mathbb{C}^n\setminus G_r^{\mathbf{b}}(\Phi)$.

We consider $\overline{K}_0=\left\{z^0+t\mathbf{b}:|t|\le\frac{r}{2L(z^0)\lambda_{\mathbf{b}}(r)}\right\}$. It does not contain zeros of $\Phi(z^0+t\mathbf{b})$, but it may contain zeros $\widetilde{c}^k=z^0+c_k^0\mathbf{b}$ of the function $\Psi(z^0+t\mathbf{b})$. Since $\Psi(z)$ is of bounded L-index in the direction $\mathbf{b}$, the set $\overline{K}_0$ by Theorem 2 contains at most $\widetilde{n}_1=\widetilde{n}_1\left(\frac{r}{2\lambda_{\mathbf{b}}(r)}\right)$ zeros c_k^0 of the function $\Psi(z^0+t\mathbf{b})$. For all $c_k^0\in\overline{K}_0$, using the definition of $Q_{\mathbf{b}}^n$, we obtain the following inequality $L(z^0+c_k^0\mathbf{b})\ge\frac{1}{\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)}L(z^0)$. Thus, every set $m_k^0=\left\{z^0+t\mathbf{b}:|t-c_k^0|\le\frac{r_1}{L(z^0+c_k^0\mathbf{b})}\right\}$ with $r_1=\frac{r}{4(\widetilde{n}_1+1)\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)\lambda_{\mathbf{b}}(r)}$ is contained in the set $s_k^0=\left\{z^0+t\mathbf{b}:|t-c_k^0|\le\frac{r_1\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)}{L(z^0)}\right\}$. The total sum of diameters of these sets does not exceed

$$\frac{2\widetilde{n}_1r_1\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)}{L(z^0)}=\frac{r}{2\lambda_{\mathbf{b}}(r)L(z^0)}\cdot\frac{\widetilde{n}_1}{(\widetilde{n}_1+1)}<\frac{r}{2\lambda_{\mathbf{b}}(r)L(z^0)}.$$

Therefore, there exists $r^*\in(0,\frac{r}{2\lambda_{\mathbf{b}}(r)})$ such that if $|t|=\frac{r^*}{L(z^0)}$, then $z^0+t\mathbf{b}\notin G_{r_1}^{\mathbf{b}}(\Psi)$, and therefore $z^0+t\mathbf{b}\notin G_{r_1}^{\mathbf{b}}(F)$. By Theorem 2 for all these points $z^0+t\mathbf{b}$ we obtain

$$\left|\frac{\partial_{\mathbf{b}}\Phi(z^0+t\mathbf{b})}{\Phi(z^0+t\mathbf{b})}\right|\le\left|\frac{\partial_{\mathbf{b}}\Psi(z^0+t\mathbf{b})}{\Psi(z^0+t\mathbf{b})}\right|+\left|\frac{\partial_{\mathbf{b}}F(z^0+t\mathbf{b})}{F(z^0+t\mathbf{b})}\right|\le(P_\Psi^*+P_F^*)L(z^0+t\mathbf{b}), \qquad (7)$$

where P_{Ψ}^{*} and P_{F}^{*} depend only on r_1, i.e. only of r. Since the function $\frac{\partial_{\mathbf{b}}\Phi(z)}{\Phi(z)}$ is analytic in $\overline{K}_0$, applying the maximum modulus principle to the function $\frac{\partial_{\mathbf{b}}\Phi(z^0+t\mathbf{b})}{\Phi(z^0+t\mathbf{b})}$ as a function of variable t, we obtain that the modulus of this function at the point $t=0$ does not exceed the maximum modulus of this function on the circle $\left\{t\in\mathbb{C}:|t|=\frac{r^*}{L(z^0)}\right\}$. It means that obtained inequality (7) also holds for z^0 instead $z^0+t\mathbf{b}$.

Thus, for arbitrary $r>0$ and $z^0\in\mathbb{C}^n\setminus G_r^{\mathbf{b}}(F)$ we have proved the first condition of Theorem 3. Above we have already shown that the second condition of Theorem 3 is true. Hence, by the mentioned theorem the function $\Phi(z)$ has bounded L-index in the direction $\mathbf{b}$.

3. Sum of functions of bounded L-index in direction. It is known that the product of entire functions of bounded L-index in direction is function with the same class ([7]). But the class of entire functions of bounded index is not closed under the addition. The corresponding example was constructed by W. Pugh (see [13, 14]). A generalization of Pugh's example for entire functions of bounded L-index in direction is proposed in [7]. The generalization is also applicable to the class $\widetilde{\mathcal{H}}_{\mathbf{b}}^{n}$.

Let us consider arbitrary hyperplane $A=\{z\in\mathbb{C}^n:\langle z,c\rangle=1\}$, where $\langle c,\mathbf{b}\rangle\neq 0$, and fixed it. Obviously that $\bigcup_{z^0\in A}\{z^0+t\mathbf{b}:t\in\mathbb{C}\}=\mathbb{C}^n$.

Let $z^0\in A$ be a given point. If $F(z^0+t\mathbf{b})\not\equiv 0$ as a function of variable $t\in\mathbb{C}$, then there exists a point $t_0\in\mathbb{C}$ such that $F(z^0+t_0\mathbf{b})\neq 0$. Thus, for every line $\{z^0+t\mathbf{b}:\ F(z^0+t\mathbf{b})\not\equiv 0\}$ we fix one point t_0 with this property. In this section, we will denote by B the union of these points $z^0+t_0\mathbf{b}$, i.e.,

$$B=\bigcup_{\substack{z^0\in A\\ F(z^0+t\mathbf{b})\not\equiv 0}}\{z^0+t_0\mathbf{b}\}.$$

Clearly, that for every $z\in\mathbb{C}^n$ there exist $z^0\in A$ and $t\in\mathbb{C}$ obeying $z=z^0+t\mathbf{b}$. Indeed, $z^0=z+\frac{1-\langle z,c\rangle}{\langle\mathbf{b},c\rangle}\mathbf{b},\quad t=\frac{\langle z,c\rangle-1}{\langle\mathbf{b},c\rangle}$.

The following propositions can be proved by analogy to [5].

Proposition 2. *Let L be a positive continuous function, and the functions F and G belong to the class $\widetilde{\mathcal{H}}_{\mathbf{b}}^{n}$ and satisfy the conditions*

1) $G(z)$ has bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ with $N_{\mathbf{b}}(G,L)=N<+\infty$;

2) there exists $\alpha \in (0,1)$ *such that for all* $z \in \mathbb{C}^n$ *and* $p \geq N+1$ $(p \in \mathbb{N})$ *one has*

$$\frac{|\partial_{\mathbf{b}}^p G(z)|}{p! L^p(z)} \leq \alpha \max\left\{\frac{|\partial_{\mathbf{b}}^k G(z)|}{k! L^k(z)} : 0 \leq k \leq N\right\}; \tag{8}$$

3) for each $z = z^0 + t\mathbf{b} \in \mathbb{C}^n$, *where* $z^0 \in A$, $z^0 + t_0\mathbf{b} \in B$ *and* $r = |t - t_0| L(z^0 + t\mathbf{b})$, *the following inequality holds*

$$\max\left\{|F(z^0 + t'\mathbf{b})| : |t' - t_0| = \frac{2r}{L(z^0 + t\mathbf{b})}\right\} \leq$$

$$\leq \max\left\{\frac{|\partial_{\mathbf{b}}^k G(z^0 + t\mathbf{b})|}{k! L^k(z^0 + t\mathbf{b})} : 0 \leq k \leq N\right\}; \tag{9}$$

4) either $(\exists c > 0)(\forall z^0 + t_0\mathbf{b} \in B)(\forall t \in \mathbb{C}, |t - t_0| L(z^0 + t\mathbf{b}) \leq 1)$:

$$\max\{|F(z^0 + t'\mathbf{b})| : |t' - t_0| = \frac{2}{L(z^0 + t\mathbf{b})}\} / |F(z^0 + t_0\mathbf{b})| \leq c < +\infty,$$

or for $L \in Q_{\mathbf{b}}^n$ $(\exists c > 0)(\forall z^0 + t_0\mathbf{b} \in B)$:

$$\max\{|F(z^0 + t'\mathbf{b})| : |t' - t_0| = \frac{2\lambda_2^{\mathbf{b}}(1)}{L(z^0 + t_0\mathbf{b})}\} / |F(z^0 + t_0\mathbf{b})| \leq c < +\infty. \tag{10}$$

Then for every $\varepsilon \in \mathbb{C}$, $|\varepsilon| \leq \frac{1-\alpha}{2c}$, *the function*

$$H(z) = G(z) + \varepsilon F(z) \tag{11}$$

has bounded L*-index in the direction* $\mathbf{b}$ *and* $N_{\mathbf{b}}(H, L) \leq N$.

Proof. The proof uses ideas from [5]. We write Cauchy's formula for the function $F(z^0 + t\mathbf{b})$ as a function of single complex variable t

$$\frac{\partial_{\mathbf{b}} F(z^0 + t\mathbf{b})}{p!} = \frac{1}{2\pi i} \int_{|t'-t| = \frac{r}{L(z^0 + t\mathbf{b})}} \frac{F(z^0 + t'\mathbf{b})}{(t' - t)^{p+1}} dt'. \tag{12}$$

For the chosen $r = |t - t_0| L(z^0 + t\mathbf{b})$ the following inequality is valid

$$\frac{r}{L(z^0 + t\mathbf{b})} = |t' - t| \geq |t' - t_0| - |t - t_0| = |t' - t_0| - \frac{r}{L(z^0 + t\mathbf{b})}.$$

Hence,

$$|t' - t_0| \leq \frac{2r}{L(z^0 + t\mathbf{b})}. \tag{13}$$

Equality (12) gives the following estimates

$$\frac{|\partial_{\mathbf{b}}^p F(z^0 + t\mathbf{b})|}{p! L^p(z^0 + t\mathbf{b})} \leq \frac{1}{2\pi L^p(z^0 + t\mathbf{b})} \cdot \frac{L^{p+1}(z^0 + t\mathbf{b})}{r^{p+1}} \times$$

$$\times \frac{2\pi r}{L(z^0 + t\mathbf{b})} \cdot \max\left\{|F(z^0 + t'\mathbf{b})| : |t' - t| = \frac{r}{L(z^0 + t\mathbf{b})}\right\} \leq$$

$$\leq \frac{1}{r^p} \max\left\{|F(z^0 + t'\mathbf{b})| : |t' - t_0| = \frac{2r}{L(z^0 + t\mathbf{b})}\right\}. \tag{14}$$

If $r=|t-t_0|L(z^0+t\mathbf{b})>1,$ then (14) implies

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\le \max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}. \qquad (15)$$

Let $r=|t-t_0|L(z^0+t\mathbf{b})\in(0;1].$ Setting $r=1$ in (12) and (13), it is possible to deduce

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\le \max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}\right\}=$$

$$=\frac{\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}\right\}}{\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}}\times$$

$$\times\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\le$$

$$\le\frac{\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|}\times$$

$$\times\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\le$$

$$\le c\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}, \qquad (16)$$

where

$$c=\sup_{z^0+t_0\mathbf{b}\in B}\ \sup_{\substack{t\in\mathbb{C},\\ |t-t_0|L(z^0+t\mathbf{b})\le 1}}\frac{\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2}{L(z^0+t\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|}\ge 1.$$

If $L\in Q,$ then $\sup\left\{\frac{L(z^0+t_0\mathbf{b})}{L(z^0+t\mathbf{b})}:|t-t_0|\le\frac{1}{L(z^0+t\mathbf{b})}\right\}\le\lambda_{\mathbf{b}}(1).$ This means that $L(z^0+t\mathbf{b})\ge\frac{L(z^0+t_0\mathbf{b})}{\lambda_{\mathbf{b}}(1)}.$ Using the obtained inequality we choose

$$c:=\sup_{z^0+t_0\mathbf{b}\in B}\frac{\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2\lambda_{\mathbf{b}}(1)}{L(z^0+t_0\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|}\ge 1$$

in (16). Taking into account (15) and (16), one has

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\le c\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\} \qquad (17)$$

for all $n \in \mathrm{N} \cup \{\mathbf{0}\}$, $r \geq 0$, $z^0 \in A$, $t \in \mathrm{C}$.

We differentiate (11) p times in the direction $\mathbf{b}$, $p \geq N+1$, and apply (8), (17) and (9) to the obtained equality

$$\frac{|\partial_{\mathbf{b}}^p H(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \leq \frac{|\partial_{\mathbf{b}}^p G(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} + \frac{|\varepsilon||\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \leq$$

$$\leq \alpha \max\left\{\frac{|\partial_{\mathbf{b}}^k G(z^0+t\mathbf{b})|}{k!L^k(z^0+t\mathbf{b})} : 0 \leq k \leq N\right\} +$$

$$+ c|\varepsilon| \max\left\{|F(z^0+t'\mathbf{b})| : |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\} \leq$$

$$\leq (\alpha + c|\varepsilon|) \max\left\{\frac{|\partial_{\mathbf{b}}^k G(z^0+t\mathbf{b})|}{k!L^k(z^0+t\mathbf{b})} : 0 \leq k \leq N\right\}. \qquad (18)$$

If $s \leq N$, then (17) is valid for $p = s$, but (8) is false. Therefore differentiation of equality (11) provides the following estimate

$$\frac{|\partial_{\mathbf{b}}^s H(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})} \geq \frac{|\partial_{\mathbf{b}}^S G(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})} - \frac{|\varepsilon||\partial_{\mathbf{b}}^s F(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})} \geq$$

$$\geq \frac{|\partial_{\mathbf{b}}^s G(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})} - c|\varepsilon| \max\left\{|F(z^0+t'\mathbf{b})| : |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\}, \qquad (19)$$

where $0 \leq s \leq N$. From (9) and (19) we get

$$\max_{0\leq s\leq N}\left\{\frac{|\partial_{\mathbf{b}}^s H(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\right\} \geq (1 - c|\varepsilon|) \max_{0\leq s\leq N}\left\{\frac{|\partial_{\mathbf{b}}^s G(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\right\}. \qquad (20)$$

If $c|\varepsilon| < 1$, then from (18) and (20) it implies

$$\frac{|\partial_{\mathbf{b}}^p H(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \leq \frac{\alpha + c|\varepsilon|}{1 - c|\varepsilon|} \max_{0\leq s\leq N}\left\{\frac{|\partial_{\mathbf{b}}^s H(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\right\} \qquad (21)$$

for $p \geq N+1$. Suppose that $\frac{\alpha + c|\varepsilon|}{1 - c|\varepsilon|} \leq 1$. Hence, $|\varepsilon| \leq \frac{1-\alpha}{2c}$.

Let $N_{\mathbf{b}}(z^0+t\mathbf{b}, L, F)$ be the L-index in direction of the function F at the point $z^0+t\mathbf{b}$, i.e., $N_{\mathbf{b}}(z^0+t\mathbf{b}, L, F)$ is the least integer m_0, for which inequality (1) holds with $z = z^0+t\mathbf{b}$.

For $|\varepsilon| \leq \frac{1-\alpha}{2c}$ validity of inequality (21) means that for any $z^0 \in A$ and for all $t \in \mathrm{C}$ such that $F(z^0+t\mathbf{b}) \neq 0$ the L-index in the direction $\mathbf{b}$ at the point $z^0+t\mathbf{b}$ does not exceed N that is $N_{\mathbf{b}}(z^0+t\mathbf{b}, F, L) \leq N$. In those points t, for which $F(z^0+t\mathbf{b}) = 0$, but $F(z^0+t\mathbf{b}) \not\equiv 0$, we will use that $H(z^0+t\mathbf{b}) = G(z^0+t\mathbf{b})$ and the function G has bounded L-index in the direction $\mathbf{b}$.

When for $z^0 \in A$ $F(z^0+t\mathbf{b}) \equiv 0$, one has $H(z^0+t\mathbf{b}) \equiv G(z^0+t\mathbf{b})$ and $N_{\mathbf{b}}(z^0+t\mathbf{b}, F, L) = N_{\mathbf{b}}(z^0+t\mathbf{b}, G, L) \leq N$. Hence, $H(z)$ is of bounded L-

index in the direction $\mathbf{b}$ with $N_{\mathbf{b}}(H,L)\le N$. This completes the proof of Theorem 2.

Every function $F\in\widetilde{\mathrm{H}}^n_{\mathbf{b}}$ with $N_{\mathbf{b}}(F,L)=0$ satisfies inequality (10). (see proof of necessity of corresponding Theorem 4 in [1]).
If $L\in Q^n_{\mathbf{b}}$, then condition 2) in Proposition 2 is always satisfied.

Proposition 3. *Let $L\in Q^n_{\mathbf{b}}$, $\alpha\in(0,1)$, F and G be functions belonging to the class $\widetilde{\mathrm{H}}^n_{\mathbf{b}}$ and obeying the conditions*

1) $G(z)$ has bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$.

2) for every $z=z^0+t\mathbf{b}\in\mathbb{C}^n$, where $z^0\in A$, $z^0+t_0\mathbf{b}\in B$, and $r=|t-t_0|L(z^0+t\mathbf{b})$ the following inequality hold

$$\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\le$$
$$\le\max\left\{\frac{|\partial^k_{\mathbf{b}}G(z^0+t\mathbf{b})|}{k!L^k(z^0+t\mathbf{b})}:0\le k\le N_{\mathbf{b}}(G_\alpha,L_\alpha)\right\}.$$

3) $$c:=\sup_{z^0+t_0\mathbf{b}\in B}\frac{\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2\lambda^{\mathbf{b}}_2(1)}{L(z^0+t_0\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|}<\infty.$$

If $|\varepsilon|\le\frac{1-\alpha}{2c}$, then the function $H(z)=G(z)+\varepsilon F(z)$ has bounded L-index in the direction $\mathbf{b}$ with $N_{\mathbf{b}}(H,L)\le N_{\mathbf{b}}(G_\alpha,L_\alpha)$, where $G_\alpha(z)=G(z/\alpha)$, $L_\alpha(z)=L(z/\alpha)$.

Proof. This proof is based on the proof of appropriate theorem for entire functions of bounded L-index in direction [5]. The condition 2) in Theorem 2 is always satisfied for $N=N_b(G_\alpha,L_\alpha)$ instead $N=N_{\mathbf{b}}(G,L)$. Indeed by Theorem 1 inequality (3) holds for the function G. Substituting $\frac{z^0}{\alpha}$, $\frac{t}{\alpha}$ and $\frac{t_0}{\alpha}$ instead z^0, t and t_0 in (3) we obtain

$$\max\left\{|G((z^0+t\mathbf{b})/\alpha)|:|t-t_0|=\frac{r_2\alpha}{L((z^0+t_0\mathbf{b})/\alpha)}\right\}\le$$
$$\le P_1\max\left\{|G((z^0+t\mathbf{b})/\alpha)|:|t-t_0|=\frac{r_1\alpha}{L((z_0+t_0\mathbf{b})/\alpha)}\right\}. \qquad (22)$$

By Theorem 1 inequality (22) means that $G_\alpha=G(z/\alpha)$ has bounded L_α-index in the direction $\mathbf{b}$ and vice versa. Hence, for all $p\ge N_{\mathbf{b}}(G_\alpha,L_\alpha)+1$ and $\alpha\in(0,1)$

$$\frac{|\partial_{\mathbf{b}}^{p}G_{\alpha}(z)|}{p!L_{\alpha}^{p}(z)}=\frac{|\partial_{\mathbf{b}}^{p}G(z/\alpha)|}{p!\alpha^{p}L^{p}(z/\alpha)}\le\max\left\{\frac{|\partial_{\mathbf{b}}^{s}G_{\alpha}(z)|}{s!L_{\alpha}^{s}(z)}:0\le s\le N_{\mathbf{b}}(G_{\alpha},L_{\alpha})\right\}=$$

$$=\max\left\{\frac{|\partial_{\mathbf{b}}^{s}G(z/\alpha)|}{s!\alpha^{s}L^{s}(z/\alpha)}:0\le s\le N_{\mathbf{b}}(G_{\alpha},L_{\alpha})\right\}.$$

Multiplying by α^{p}, we deduce

$$\frac{|\partial_{\mathbf{b}}^{p}G(z/\alpha)|}{p!L^{p}(z/\alpha)}\le\max\left\{\frac{\alpha^{p-s}|\partial_{\mathbf{b}}^{s}G(z/\alpha)|}{s!L^{s}(z/\alpha)}:0\le s\le N_{\mathbf{b}}(G_{\alpha},L_{\alpha})\right\}\le$$

$$\le\alpha\max\left\{\frac{|\partial_{\mathbf{b}}^{s}G(z/\alpha)|}{s!L^{s}(z/\alpha)}:0\le s\le N_{\mathbf{b}}(G_{\alpha},L_{\alpha})\right\}.$$

In view of arbitrariness z, the last inequality yields (8).

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СУМА І ДОБУТОК ФУНКЦІЙ ОБМЕЖЕНОГО *L*-ІНДЕКСУ ЗА НАПРЯМКОМ, ЯКІ ГОЛОМОРФНІ НА ЗРІЗКАХ ЗА ТИМ САМИМ НАПРЯМКОМ

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У статті досліджено голоморфні на зрізках функції $F:\mathbb{C}^n \to \mathbb{C}$ *обмеженого* L*-індексу за напрямком, себто ці функції є цілими на кожній зрізці* $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$ *для довільного* $z^0 \in \mathbb{C}^n$ *та для фіксованого напрямку* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$*, та* $(\exists m_0 \in \mathbb{Z}_+)$ $(\forall m \in \mathbb{Z}_+)$ $(\forall z \in \mathbb{C}^n)$ *для яких виконується наступна нерівність*

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m! L^m(z)} \le \max_{0\le k\le m_0} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)},$$

де $\mathbf{L}:\mathbb{C}^n \to \mathbb{R}_+$ *- додатна неперервна функція,* $\partial_{\mathbf{b}} F(z) = \frac{d}{dt} F(z+t\mathbf{b})|_{t=0}$, $\partial_{\mathbf{b}}^p F = \partial_{\mathbf{b}}(\partial_{\mathbf{b}}^{p-1} F)$ *для* $p \ge 2$. *Об'єктами дослідження є сума та добуток функцій з цього класу. Знайдено достатні умови, які забезпечують обмеженість* L*-індексу за тим самим напрямком для цих операцій. Зокрема, множення функцій з цього класу є замкненою операцією.*

Ключові слова: *обмежений індекс, обмежений L-індекс за напрямком, функція зрізки, голоморфна функція, обмежений l-індекс, сума.*