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CONVERGENCE OF GAUSS CONTINUED FRACTION FOR THE RATIO OF HYPERGEOMETRIC FUNCTIONS IN $\mathbb{Q}_p$

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The conditions of convergence of Gauss continued fraction to the ratio of hypergeometric functions in the field of p-adic numbers are established.
Key words: *hypergeometric functions of Gauss, continued fraction of Gauss, p-adic numbers.*

1. Introduction and formulation of main results

The following fraction is called Gauss continued fraction [1, 2]

$$1+\mathop{D}_{n=1}^{\infty}\frac{a_n z}{1}, \qquad z\in\mathbf{C}, \tag{1}$$

where

$$a_{2n+1}=-\frac{(a+n)(c-b+n)}{(c+2n)(c+2n+1)},\ n\ge 0,\ a_{2n}=-\frac{(b+n)(c-a+n)}{(c+2n-1)(c+2n)},\ n\ge 1, \tag{2}$$

and a,b,c are any complex numbers, such that $c\notin\{0;-1;-2;\dots\}$. Let us notice that if at least one of the numbers a,b belongs to the set $\{0;-1;-2;\dots\}$, then the fraction (1) reduces to the ratio of polynomials.

The fraction (1) arises from the expansion of the ratio of Gauss hypergeometric functions

$$\frac{F(a,b;c;z)}{F(a,b+1;c+1;z)} \tag{3}$$

into continued fraction [2]. Let us recall [3] that Gauss function $F(a,b;c;z)$ is given inside the disk $\{z\in\mathbf{C}:|z|<1\}$ by the sum of Gauss hypergeometric series

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (4)$$

where $a,b,c \in \mathbf{C}$, $c \notin \{0;-1;-2;\ldots\}$, $(\cdot)_n$ are Pochhammer symbols:

$$(a)_0 = 1,\ (a)_n = a(a+1)\cdot\ldots\cdot(a+n-1),\ n \in \mathbf{N}.$$

In similar way from (4) we can obtain following equalities

$$F(a,b;c;z) = F(a,b+1;c+1;z) - \frac{a(c-b)}{c(c+1)} zF(a+1,b+1;c+2;z), \quad (5)$$

$$F(a,b+1;c+1;z) = F(a+1,b+1;c+2;z) - \\ - \frac{(b+1)(c-a+1)}{(c+1)(c+2)} zF(a+1,b+2;c+3;z), \quad (6)$$

From equalities (5), (6) we obtain the following recurrent relations

$$w_n(z) = 1 + \frac{a_{n+1} z}{w_{n+1}(z)}, \quad n \ge 0, \quad (7)$$

where

$$w_{2n+1}(z) = \frac{F(a+n,b+n+1;c+2n+1;z)}{F(a+n+1,b+n+1;c+2n+2;z)}, \quad n \ge 0,$$
$$w_{2n}(z) = \frac{F(a+n,b+n;c+2n;z)}{F(a+n,b+n+1;c+2n+1;z)}, \quad n \ge 1, \quad (8)$$

and a_n, $n \ge 1$, are defined by equations (2). Then from equalities (7), (8) we obtain

$$\frac{F(a,b;c;z)}{F(a,b+1;c+1;z)} = 1 + \cfrac{a_1 z}{1 + \cfrac{a_2 z}{\ddots + \cfrac{\ddots}{1 + \cfrac{a_n z}{w_n(z)}}}}, \ n \ge 2.$$

So we get a continued fraction expansion of the ratio (3) (see [2])

$$\frac{F(a,b;c;z)}{F(a,b+1;c+1;z)} \sim 1 + \underset{n=1}{\overset{\infty}{D}} \frac{a_n z}{1}.$$

The sequence of functions

$$f_0(z) = 1, \qquad f_n(z) = 1 + \underset{k=1}{\overset{n}{D}} \frac{a_k z}{1}, \quad n \in \mathbf{N}, \quad (9)$$

is called the sequence of approximants of fraction (1).

The Gauss fraction (1) is said to converge (uniformly) to the function $G(z)$ in the set M, if the sequence of its approximants $\{f_n(z)\}_{n=0}^{\infty}$ converges (uniformly) on M to $G(z)$ as $n \to \infty$. The interesting question is: for which a,b,c and for which value z fraction (9) converges to the relation (3).

In the work [2] it was established that for $a,b,c \in \mathbf{C}$, $c \notin \{0;-1;-2;\dots\}$, Gauss continued fraction (1) converges to the ratio (3) in the cut plane $P = \{ z \in C : | arg(1-z) | < \pi \}$ and convergence is uniformly on every compact subset of $\{z \in P : G(z) \neq \infty\}$, where $G(z) = \lim\limits_{n \to \infty} f_n(z)$.

In present work the results of the work [2] are transferred to the case when the parameters a,b,c,z of the Gauss continued fraction (1) are p-adic numbers and the convergence of sequence of approximants (9) is considered in the p-adic norm. The main result of this work consists in the following propositions:

Theorem 1. *Let* $a,b,c \in \mathbf{Q}_p$ *be such that*

$$|a|_p \neq |c|_p,\ |b|_p \neq |c|_p,\ \min\{|a|_p, |b|_p\} > 1,\ |c|_p > \max\{|a|_p, |b|_p\}.$$

Then fraction (1) *uniformly converges in the* p*-adic disk* $\{ z \in \mathbf{Q}_p : |z|_p < 1 \}$.

Theorem 2. *Let* $a,b,c \in \mathbf{Q}_p$ *be such that*

$$|a|_p \neq |c|_p,\ |b|_p \neq |c|_p,\ \min\{|a|_p, |b|_p\} > 1,\ |c|_p > \max\{|a|_p, |b|_p, |ab|_p\}.$$

Then Gauss fraction uniformly converges in the p*-adic disk* $\left\{z \in \mathbf{Q}_p : |z|_p < p^{1/(1-p)}\right\}$ *to the ratio (3).*

2. Basic concepts of *p*-adic numbers

In order to prove Theorems 1, 2 let us recall some concepts of the theory of *p*-adic numbers [4]. Let us define the *p*-adic norm in the set of rational numbers $\mathbf{Q}$ by the rule

$$|0|_p = 0, \quad |x|_p = \frac{1}{p^{\operatorname{ord}_p x}}, \qquad x \in \mathbf{Q} \setminus \{0\},$$

where *p* is the prime number, and where the *p*-adic ordinal $\operatorname{ord}_p x$ of the rational number *x* is defined by means of the equality

$$\operatorname{ord}_p x = \begin{cases} \max\{m \in Z_+ : x \equiv 0 (\operatorname{mod} p^m)\}, & \text{if} \quad x \in \mathbf{Z},\ x \neq 0, \\ \operatorname{ord}_p a - \operatorname{ord}_p b, & \text{if} \quad x = \dfrac{a}{b}, \quad a,b \in \mathbf{Z} \setminus \{0\}. \end{cases}$$

The field of *p*-adic numbers, denoted by the symbol $\mathbf{Q}_p$, is defined as the completion of the field of rational numbers $\mathbf{Q}$ with respect to the *p*-adic norm introduced above. For the p-adic norm the strengthened triangle inequality holds, namely

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

This inequality implies *the principle of isosceles triangle* [4] for the field $\mathbf{Q}_p$, which consists in that for any $x, y \in \mathbf{Q}_p$ the alternative holds: either $|x|_p = |y|_p$, or $|x \pm y|_p \leq \max\{|x|_p, |y|_p\}$, if $|x|_p \neq |y|_p$.

3. Properties of the partial numerators

Now we shall obtain properties of a_n, $n \ge 1$, defined by the equality (2). Let us denote: $D(r) = \{z \in \mathbf{Q}_p : |z|_p < r\}$, $r > 0$.

Lemma 1. *If* $a,b,c \in \mathbf{Q}_p$, $|a|_p \ne |c|_p$, $|b|_p \ne |c|_p$, $\min\{|a|_p, |b|_p\} > 1$, $|c|_p > \max\{|a|_p, |b|_p\}$, *then following equalities hold:*

$$|a_n|_p \le \max\{|a|_p, |b|_p\} / |c|_p < 1, \quad n \ge 1.$$

Proof. As $|n|_p \le 1$ for any $n \in \mathbf{N}$, then from the conditions of Lemma 1 together with the principle of the isosceles triangle [4] it follows that

$$|a+n|_p = |a|_p, \ |b+n|_p = |b|_p, \ |c+n|_p = |c|_p, \quad n \in \mathbf{N},$$
$$|c-a+n|_p = |c|_p, \ |c-b+n|_p = |c|_p, \quad n \in \mathbf{N}. \tag{10}$$

From the inequality $|c|_p > \max\{|a|_p, |b|_p\}$ together with the relations (2), (10) we obtain that the following relations hold

$$|a_{2n+1}|_p = |a|_p / |c|_p \le \max\{|a|_p, |b|_p\} / |c|_p < 1, \quad n \ge 0,$$
$$|a_{2n}|_p = |b|_p / |c|_p \le \max\{|a|_p, |b|_p\} / |c|_p < 1, \quad n \ge 1.$$

Lemma is proved.

4. Properties of the canonical numerators and denominators

Let us define *p*-adic norms of the canonical numerators and denominators of the Gauss fraction. Let us remark that the recurrence sequences of functions $\{A_n(z)\}_{n=0}^{\infty}$, $\{B_n(z)\}_{n=0}^{\infty}$, which are defined from the equalities

$$A_0(z) = 1, \quad A_1(z) = a_1 z + 1, \quad B_0(z) = 1, \quad B_1(z) = 1,$$
$$A_n(z) = A_{n-1}(z) + a_n z A_{n-2}(z), \ B_n(z) = B_{n-1}(z) + a_n z B_{n-2}(z), \ \ n \ge 2, \tag{11}$$

are the canonical numerators and denominators of approximants of the Gauss fraction, so that

$$f_n(z) = A_n(z) / B_n(z), \ n \in \mathbf{N} \cup \{0\}.$$

Lemma 2. *Let* $a,b,c \in \mathbf{Q}_p$, $|a|_p \ne |c|_p$, $|b|_p \ne |c|_p$, $\min\{|a|_p, |b|_p\} > 1$, $|c|_p > \max\{|a|_p, |b|_p\}$. *If* $z \in D(1)$ *then*

$$|A_n(z)|_p = 1, \quad |B_n(z)|_p = 1, \ \ n \in \mathbf{N} \cup \{0\}. \tag{12}$$

Proof. We shall apply the method of mathematical induction on n. It is obvious that $|A_0(z)|_p = 1$, $|B_0(z)|_p = 1$, $|B_1(z)|_p = 1$. From Lemma 1 for all $z \in D(1)$ we obtain $|a_1 z|_p < |a_1|_p < 1$, therefore from the principle of isosceles triangle we obtain $|A_1(z)|_p = 1$. Thus the equalities (12) are true for $n = 0,1$ and the base of induction is established.

We assume that equalities (12) are true for all $n<k$, where $k\geq 3$. Then from Lemma 1 and from the inductive hypothesis it follows that for all $z\in D(1)$

$$|A_{k-1}(z)|_p=1,\ |a_k zA_{k-2}(z)|_p=|a_k z|_p<|a_k|_p<1,$$

$$|B_{k-1}(z)|_p=1,\ |a_k zB_{k-2}(z)|_p=|a_k z|_p<|a_k|_p<1.$$

From these relations together with the recurrent relations (11) and the principle of isosceles triangle we obtain

$$|A_k(z)|_p=\max\{|A_{k-1}(z)|_p,|a_k zA_{k-2}(z)|_p\}=1,$$

$$|B_k(z)|_p=\max\{|B_{k-1}(z)|_p,|a_k zB_{k-2}(z)|_p\}=1,$$

what means that the equalities (12) hold for $n=k$. Therefore, the step of induction is obtained. Lemma is proved.

5. The convergence of sequence of approximants

Let us establish conditions of convergence of the sequence $\{f_n(z)\}_{n=0}^{\infty}$, defined by formula (9).

Lemma 3. *Let* $a,b,c\in\mathbf{Q}_p$, $|a|_p\neq|c|_p$, $|b|_p\neq|c|_p$, $\min\{|a|_p,|b|_p\}>1$, $|c|_p>\max\{|a|_p,|b|_p\}$. *If* $z\in D(1)$ *then for all* $n\in\mathbf{N}$

$$\left|f_n(z)-f_{n-1}(z)\right|_p=|a_1|_p\cdot\ldots\cdot|a_n|_p\cdot|z|_p^n. \qquad (13)$$

Proof. We shall use the method of mathematical induction on n. For $n=1$ we have

$$\left|f_1(z)-f_0(z)\right|_p=\left|\frac{A_1(z)}{B_1(z)}-\frac{A_0(z)}{B_0(z)}\right|_p=\left|\frac{A_1(z)B_0(z)-A_0(z)B_1(z)}{B_1(z)B_0(z)}\right|_p=|a_1 z|_p.$$

Let us assume that formula (13) is true for $n=k$, $k\geq 1$. Now we will prove that it is true for $n=k+1$. In fact, on the basis of Lemma 2, $\left|B_n(z)\right|_p=1$ for all $n\in\mathbf{N}$, so that

$$\left|f_{k+1}(z)-f_k(z)\right|_p=\left|\frac{A_{k+1}(z)}{B_{k+1}(z)}-\frac{A_k(z)}{B_k(z)}\right|_p=$$

$$=\left|\frac{A_{k+1}(z)B_k(z)-A_k(z)B_{k+1}(z)}{B_{k+1}(z)B_k(z)}\right|_p=\left|A_{k+1}(z)B_k(z)-A_k(z)B_{k+1}(z)\right|_p.$$

By applying to the functions $A_{k+1}(z)$, $B_{k+1}(z)$ the recurrent relations (11) in complience with the induction assumptions and according to Lemmas 1, 2, we obtain that

$$|f_{k+1}(z)-f_k(z)|_p=\left|A_k(z)B_k(z)+a_{k+1}zA_{k-1}(z)B_k(z)-\right.$$

$$\left.-A_k(z)B_k(z)-a_{k+1}zA_k(z)B_{k-1}(z)\right|_p=$$

$$=\left|a_{k+1}z(A_{k-1}(z)B_k(z)-A_k(z)B_{k-1}(z))\right|_p=$$

$$=|a_{k+1}z|_p|f_k(z)-f_{k-1}(z)|_p=|a_1|_p\cdot\ldots\cdot|a_{k+1}|_p\cdot|z|_p^{k+1}.$$

Lemma is proved.

Proof of Theorem 1. Based on the assumptions of the Theorem and of Lemmas 1, 3 it follows that for any $n,m\in\mathbf{N}$, $m>n$, and $z\in D(1)$ the next estimates are true:

$$|f_n(z)-f_m(z)|_p\le\max_{n+1\le j\le m}|f_j(z)-f_{j-1}(z)|_p<(\max\{|a|_p,|b|_p\}/|c|_p)^n.$$

From inequality $|c|_p>\max\{|a|_p,|b|_p\}$ the fundamentality of sequence (9) in $\mathbf{Q}_p$ follows and so its convergence in $\mathbf{Q}_p$ follows too.

6. Convergence of the sequence of approximants to the ratio of hypergeometric functions

The Theorem 1 bring us to the fact that in the circle $D(1)$ there exists a function $f:D(1)\to\mathbf{Q}_p$, which is the point limit of the sequence (9):

$$f(z):=\lim_{n\to\infty}f_n(z),\ z\in D(1).$$

From Lemma 2 it follows that the image of the map $f(z)$ in fact is a subset of unit circle $\{z\in\mathbf{Q}_p:|z|_p=1\}$.

Let us establish the requirements for the parameters $a,b,c\in\mathbf{Q}_p$ for which the function $f(z)$ equals the ratio (3).

Lemma 4. *Let* $a,b,c\in\mathbf{Q}_p$ *and* $\min\{|a|_p,|b|_p\}>1$, $|c|_p>|ab|_p$. *If* $z\in D(p^{1/(1-p)})$, *then* $|F(a,b;c;z)|_p=1$ *for all* $z\in D(p^{1/(1-p)})$.

Proof. It is known [4] that

$$|1/n!|_p\le p^{n/(p-1)},\qquad n\in\mathbf{N}.\tag{14}$$

From the conditions of Lemma and the formulas (10), (14) we see that for all $n\ge1$ and $z\in D(p^{1/(1-p)})$

$$\left|\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\right|_p\le\frac{|ab|_p^n}{|c|_p^n}p^{n/(p-1)}|z|_p^n\le\frac{|ab|_p^n}{|c|_p^n}.\tag{15}$$

From the inequality $|c|_p>|ab|_p$ together with the estimates (15) the convergence of series (4) follows, so from the principle of isosceles triangle it follows that

$$|F(a,b;c;z)|_p=\max\left\{1;\sup_{n\ge1}\left\{\left|\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\right|_p\right\}\right\}=1.$$

Lemma is proved.

Lemma 5. *Let* $a,b,c \in \mathbf{Q}_p$, $|a|_p \neq |c|_p$, $|b|_p \neq |c|_p$, $\min\{|a|_p, |b|_p\} > 1$, $|c|_p > \max\{|a|_p, |b|_p, |ab|_p\}$. *If* $z \in D(p^{1/(1-p)})$ *then*

$$\left| f_n(z) - \frac{F(a,b;c;z)}{F(a,b+1;c+1;z)} \right|_p = |a_1|_p \cdot \ldots \cdot |a_{n+1}|_p \cdot |z|_p^n, \; n \in \mathbf{N}. \qquad (16)$$

Proof. Let us prove that for all $n \in \mathbf{N}$ the following formula is true

$$\frac{F(a,b;c;z)}{F(a,b+1;c+1;z)} = \frac{a_{n+1}zA_{n-1}(z) + w_{n+1}(z)A_n(z)}{a_{n+1}zB_{n-1}(z) + w_{n+1}(z)B_n(z)}, \qquad (17)$$

where $w_n(z)$, $n \in \mathbf{N}$, are defined by the equation (8). Let us use the method of mathematical induction on n. For $n = 1$ we have

$$\frac{F(a,b;c;z)}{F(a,b+1;c+1;z)} = 1 + \frac{a_1 z}{w_1(z)}.$$

Let us assume that the formula (17) it true for $n < k$. Now we shall prove that it holds for $n = k$. From the induction assumptions we obtain

$$\frac{F(a,b;c;z)}{F(a,b+1;c+1;z)} = \frac{a_k zA_{k-2}(z) + w_k(z)A_{k-1}(z)}{a_k zB_{k-2}(z) + w_k(z)B_{k-1}(z)}.$$

Since $w_k(z) = 1 + \frac{a_{k+1}z}{w_{k+1}(z)}$ (see formula (7)), then

$$\frac{F(a,b;c;z)}{F(a,b+1;c+1;z)} = \frac{a_k zA_{k-2}(z) + \left(1 + \frac{a_{k+1}z}{w_{k+1}(z)}\right)A_{k-1}(z)}{a_k zB_{k-2}(z) + \left(1 + \frac{a_{k+1}z}{w_{k+1}(z)}\right)B_{k-1}(z)} =$$

$$= \frac{a_{k+1}zA_{k-1}(z) + w_{k+1}(z)A_k(z)}{a_{k+1}zB_{k-1}(z) + w_{k+1}(z)B_k(z)}.$$

Therefore from the formula (17) it follows that

$$\left| f_n(z) - \frac{F(a,b;c;z)}{F(a,b+1;c+1;z)} \right|_p =$$

$$= \left| \frac{a_n zA_{n-2}(z) + A_{n-1}(z)}{a_n zB_{n-2}(z) + B_{n-1}(z)} - \frac{a_n zA_{n-2}(z) + w_n(z)A_{n-1}(z)}{a_n zB_{n-2}(z) + w_n(z)B_{n-1}(z)} \right|_p =$$

$$= \frac{|f_{n-1}(z) - f_{n-2}(z)|_p |a_n z|_p |1 - w_n(z)|_p}{|a_n zB_{n-2}(z) + B_{n-1}(z)|_p |a_n zB_{n-2}(z) + w_n(z)B_{n-1}(z)|_p}.$$

Since from Lemma 4 and formula (8) it follows that $|w_n(z)|_p = 1$ for all $n \in \mathbf{N}$, then from Lemmas 2, 3 and formula (7) we obtain (16).

Lemma is proved.

Proof of Theorem 2. Since from the assumptions of the Theorem 2 and of Lemmas 1, 5 it follows that for any $n \in \mathbf{N}$ and $z \in D(p^{1/(1-p)})$ it is true that

$$\left| f_n(z) - \frac{F(a,b;c;z)}{F(a,b+1;c+1;z)} \right|_p \leq (\max\{|a|_p, |b|_p\}/|c|_p)^{n+1}.$$

From this inequality and inequality $|c|_p > \max\{|a|_p, |b|_p\}$ it follows that the sequence of functions (9) converges to the ratio (3) in $\mathbf{Q}_p$.

Література

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$$1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot2\cdot\gamma\cdot(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot2\cdot3\cdot\gamma\cdot(\gamma+1)\cdot(\gamma+2)}x^3+etc.,$$

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ЗБІЖНІСТЬ НЕПЕРЕРВНОГО ДРОБУ ГАУССА ДО ВІДНОШЕННЯ ГІПЕРГЕОМЕТРИЧНИХ ФУНКЦІЙ В $\mathbb{Q}_p$

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Встановлено умови збіжності неперервного дробу Гаусса до відношення значень гіпергеометричних функцій Гаусса в полі p-адичних чисел.

Ключові слова: *гіпергеометрична функція Гаусса, неперервний дріб Гаусса, p-адичні числа.*